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NONLINEAR SYSTEM IDENTIFICATION STUDY PART II COMPUTATIONAL COMPLEXITY STUDY

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The computational complexity of a nonlinear system identification technique is evaluated in this report. The identification technique uses a "black box" approach requiring measurements only at system input and output terminals and is applicable to weakly nonlinear systems whose behavior is adequately characterized by a finite Volterra Series.

The computational aspects of the technique are evaluated in terms of the complexity of the calculations and the complexity of the system being implemented.

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Techniques for reducing the order of the second order response are investigated. These techniques include restricted frequency range, integration time control, and dominant pole concepts. The class of systems to which the technique can be applied is evaluated.

EVALUATION

The process of system identification consists of postulating a valid analytical model for the system under consideration and performing tests on the system to completely specify or "identify" the parameters which describe the system analytical model. For example, a linear system is completely characterized by its impulse response, h(t). The system identification process for this linear system analytical model consists of any procedure that completely determines h(t). The present consideration in the area of nonlinear system identification is the derivation of a valid analytical model for the nonlinear system under consideration.

The identification procedure successfully studied is a black box technique where only input and output terminal measurements of the nonlinear system are used. The identification technique is applicable to a broad class of weakly nonlinear systems whose response can be characterized by a finite Volterra series. The identification procedure involves processing the input and output responses of a nonlinear system to obtain a set of linearly independent equations which uniquely define the parameters of a functional form of the second-order impulse response. Theoretically, the proposed identification technique represents a significant improvement over existing identification techniques because of its black box formulation. The intent of the study was to determine where this identification technique can be practically implemented and maintain an advantage over existing tecniques. To these ends, the practical implementation constraints have been developed, quantified and assessed for three candidate measurement configurations. The robustness of the technique to nonlinear circuits with many and/or repeated poles is the subject of Part II of this final report.

The study effort successfully accomplished, in two parts, a Part I on implementation feasibility to determine the practical methods and constraints of implementing three candidate measurement configurations - digital, analog and hybrid. The second part of the study effort successfully focused on the numerical computation complexity aspect of the identification technique processing to determine the class(es) of nonlinear systems for which the technique can be practically applied. The primary computational complexity arises from the required matrix inversions for the residue evaluations. Toward the goal of alleviating these difficulties, matrix scaling, band limited approaches, single exponential inputs (multiple input times) and dominant pole concepts were also developed, quantified and assessed in the successful pursuit of the overall study objectives.

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TECHNICAL REPORT SUMMARY FOR RADC-TR-79-
NONLINEAR SYSTEM IDENTIFICATION STUDY
PART II. COMPUTATIONAL COMPLEXITY STUDY

A. STUDY OBJECTIVES

The basic objective of this study effort is to evaluate the practical feasibility of a nonlinear system identification technique. The identification procedure studied is a black box technique where only input and output terminal measurements of the nonlinear system are used. The identification technique is applicable to a broad class of weakly nonlinear systems whose response can be characterized by a finite Volterra series. The identification procedure involves processing the input and output responses of a nonlinear system to obtain a set of linearly independent equations that uniquely define the parameters of a functional form of the second-order impulse response. Theoretically, the proposed identification technique represents a significant improvement over existing identification techniques because of its black box formulation. The intent of the study is to determine if this identification technique can be practically implemented and maintain an advantage over existing techniques.

The study effort is divided into two parts:

Part I — An implementation feasibility study to determine practical methods of implementing the measurement scheme - both digital and analog - and to evaluate the requirements for the components of the measurement scheme.

Part II — A computational complexity study of the identification technique processing to determine the class of nonlinear systems to which the technique can be practically applied.

This final report represents the results of Part II of the study effort - the computational complexity study.

B. SUMMARY OF RESULTS AND CONCLUSIONS

This part of the study effort focused on identifying the computational limitations of the identification technique that restrict its application to practical systems and on developing methods of easing these limitations.

The primary computational limitations of the identification technique arise from the required matrix inversions necessary to evaluate the system residues. The dynamic range of the matrix entries increases as the matrix size increases and these entries can violate the dynamic range constraints of typical general-purpose computers even for moderate size systems. This problem is complicated further when the linear system transfer function is wide band.

Three approaches are suggested for alleviating these computational problems:

(1) Matrix scaling

(2) Reduction of the order of $Y_2(s)$

(3) Computational scaling for the total identification technique.

The third approach was not addressed in detail in the study but requires a continuous scaling of matrix inversion operations to take advantage of the total dynamic range of the digital computer.

Matrix scaling techniques were investigated on the basis of known algorithms. The results of this investigation demonstrate that these techniques offer some relief of the computational problems but do not substantially increase the applicability of the identification technique.

A primary conclusion of this study is that the order of the second order response must be reduced as much as possible while still maintaining the integrity of the identification technique. This approach dictates a more extensive measurement process for the identification technique. Several approaches to accomplish this reduction of the order of $Y_2(s)$ are postulated in this report. These approaches include:

(1) Restricted Frequency Approaches - including use of a low-pass filter at the system output, and appropriate selection of integration time.

(2) Single Exponential Input - the input signal consists of a single exponential function $x(t) = e^{-\alpha_i t}$ instead of

$$\sum_{i=1}^{N} e^{-\alpha_i t}$$

The input is applied N times (changing α_i) and an appropriate set of measurements is taken. The identification process is essentially repeated N times to generate the required set of linearly independent equations.

(3) Dominant Pole Concept - The linear transfer function is modeled by a lower order transfer function where the dominant poles are used in the transfer function model.

The computational problem introduced by a wide-band system, i.e., a near singular matrix to be inverted, can be avoided by the following method. The poles of $Y_2(s)$ of the form $\lambda_i + \lambda_j \approx \lambda_j$ are modeled as single poles at λ_j and the residues are combined to obtain the total residue. A modification of the identification technique allows identification of the $A_{k_1 k_2}$ quantities.

This has been demonstrated for two pairs of poles of the form $\lambda_i + \lambda_j \approx \lambda_j$.

The issue of the isolation of the second-order response from the total system response was also addressed in this study. It was shown that it is not necessary to isolate $y_2(t)$ from $y_1(t) + y_2(t)$ to identify the linear and second-order impulse responses. However, it was also shown that $y_2(t)$ must be isolated from third and higher-order system responses in order to identify the second-order impulse response, $h_2(t_1,t_2)$.

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INTRODUCTION

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Part I — An implementation feasibility study to determine practical methods of implementing the measurement scheme - both digital and analog - and to evaluate the requirements for the components of the measurement scheme.

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This final report represents the results of Part II of the study effort - the computational complexity study. The implementation feasibility study results were presented in Part I of this final report (Reference 1).

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This part of the study effort focused on identifying the computational limitations of the identification technique that restrict its application to practical systems and on developing methods of easing these limitations.

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Matrix scaling techniques were investigated on the basis of the algorithms developed in Reference 2. The results of this investigation demonstrate that these techniques offer some relief of the computational problems but do not substantially increase the applicability of the identification technique.

A primary conclusion of this study is that the order of the second order response must be reduced as much as possible while still maintaining the integrity of the identification technique. In many instances, this approach dictates a more extensive measurement process than originally required (Reference 1). Several approaches to accomplish this reduction of the order of $Y_2(s)$ are postulated in this report. These approaches include:

(1) Restricted Frequency Approaches - including use of a low-pass filter at the system output, and appropriate selection of integration time

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The issue of the isolation of the second-order response from the total system response was also addressed in this study. It was shown that it is not necessary to isolate $y_2(t)$ from $y_1(t) + y_2(t)$ to identify the linear and second-order impulse responses. However, it was also shown that $y_2(t)$ must be isolated from third and higher-order system responses in order to identify the second-order impulse response, $h_2(t_1,t_2)$.

SECTION II

IDENTIFICATION TECHNIQUE

A. IDENTIFICATION TECHNIQUE BACKGROUND

The basic objective of this study (Part II) is to investigate the computational complexity aspects of an identification technique for nonlinear systems. The identification technique is briefly reviewed in this section. The identification technique is described in detail in the Part I Final Report (Reference 1) and is based on the analysis presented in Reference 3. This technique is a "black box" procedure in that only measurements at the system input and output terminals are required. The identification technique is applicable to a class of weakly nonlinear systems whose behavior is adequately characterized in terms of a finite Volterra functional series given by

$$y(t) = \sum_{n=1}^{\bar{N}} y_n(t) = \sum_{n=1}^{\bar{N}} \underline{\int} h_n(\tau_1, \ldots, \tau_n) \prod_{p=1}^{n} x(t - \tau_p)\, d\tau_p \quad (1)$$

where

$y_n(t)$ is the n order portion of the response

$\underline{\int}$ denotes an n-fold integration from $-\infty$ to $+\infty$

$\prod_{p=1}^{n}$ denotes an n-fold product.

$\bar{N}$ denotes the number of terms in the infinite volterra series.

The n^{th}-order Volterra kernel $h_n(\tau_1, \ldots, \tau_n)$ can be referred to as the n^{th}-order nonlinear impulse response (Reference 4). In actuality, the nonlinear impulse responses may not be identically zero above order $\bar{N}$. However, the finite sum of equation (1) implies that higher-order terms contribute negligibly to the output.

The identification technique developed in Reference 3 is designed to identify the parameters of closed-form expressions for the nonlinear impulse responses, $h_n(t_1, t_2, \ldots, t_n)$,

$n = 1, 2, \ldots, \bar{N}$. The analysis presented in Reference 3 demonstrates how the technique identifies the parameters of $h_1(t)$, $h_2(t_1, t_2)$ and $h_3(t_1, t_2, t_3)$. On the basis of this analysis, it is believed that the technique is extendable to identification of higher order nonlinear impulse responses ($\bar{N} \geq 4$). This study (Part II) is concerned with the computational aspects of the identification of the linear and second-order nonlinear impulse responses, $h_1(t)$, $h_2(t_1,t_2)$.

It has been shown (Reference 3) that, when the linear incremental model of a nonlinear system is described by

$$h_1(t) = \begin{cases} \sum_{i=1}^{N} R_i e^{\lambda_i t} & , t \geq 0 \\ 0 & , t < 0 \end{cases} \tag{2}$$

where Re $\{\lambda_i\} \leq 0$ and it is assumed that the λ_i are distinct, the second-order nonlinear impulse response can be expressed in the symmetrical form (Reference 3):

$$h_2(t_1,t_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_1+a_{k_2}t_2} U(t_2 - t_1) + \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_2+a_{k_2}t_1} U(t_1 - t_2) \tag{3}$$

where

$$M = N^2 + 1. \tag{4}$$

$$U(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases} \tag{5}$$

and where the natural frequencies in equation (3) are related to those in equation (2) according to:

$$
\begin{aligned}
&a_1 = \lambda_1,\ a_2 = \lambda_2, \ldots, a_N = \lambda_N \quad , \\
&a_{N+1} = \lambda_1 - \lambda_1 = 0,\ a_{N+2} = \lambda_1 - \lambda_2, \ldots,\ a_{2N} = \lambda_1 - \lambda_N \\
&a_{2N+1} = \lambda_2 - \lambda_1,\ a_{2N+2} = \lambda_2 - \lambda_3, \ldots,\ a_{3N-1} = \lambda_2 - \lambda_N \\
&a_{N^2-N+3} = \lambda_N - \lambda_1,\ a_{N^2-N+4} = \lambda_N - \lambda_2, \ldots, \\
&\qquad a_{N^2+1} = \lambda_N - \lambda_{N-1}. \qquad (6)
\end{aligned}
$$

The ordering of the a_{k1} terms in equation (3) assumes all the factors $\lambda_i - \lambda_j$ to be distinct, such that $\lambda_i - \lambda_j \neq \lambda_k$ for any $i, j, k = 1, \ldots, N$. Also, the zero entry that results from $\lambda_i - \lambda_j$ when $i = j$ is included only once as the entry a_{N+1}. In addition, it is readily shown that (Reference 3)

$$
A_{k_1 k_2} = A_{k_2 k_1} \qquad \text{for } k_1, k_2 \leq N \qquad (7)
$$

and that the coefficients of terms in equation(3) having the form

$$
e^{(\lambda_i - \lambda_j)t_1 + \lambda_i t_2} \quad , \qquad i \neq j
$$

are identically zero.

The identification technique identifies the parameters of $h_2(t_1, t_2)$ as represented in equation (3).

B. IDENTIFICATION TECHNIQUE DESCRIPTION

The functional form for $h_2(t_1, t_2)$ established in equation (3) implies that the identification of $h_2(t_1, t_2)$ reduces to identification of the parameters a_{k1}, a_{k2}, $A_{k1\,k2}$ and N. However, equations (4) and (6) show that a_{k1}, a_{k2} and N can be determined once the linear impulse response is known. Therefore, the task of identifying these parameters reduces to the task of identifying $h_1(t)$. The problem of identifying the coefficients $A_{k_1 k_2}$ still remains.

The identification process separates into two distinct steps: (1) identification of $h_1(t)$; and (2) identification of the $A_{k1\,k2}$ quantities of $h_2(t_1, t_2)$. These two steps are considered below.

1. Identification of the Linear Impulse Response, $h_1(t)$.

The first step in the identification of $h_1(t)$, the linear impulse response of a nonlinear system, is to excite the system with an input amplitude such that the output is linear. The amplitude of this signal can be determined by exciting the system with a sinusoidal signal of amplitude A and performing a spectral analysis of the resultant response. Amplitude A is then adjusted until the amplitude level of the harmonic frequencies of the output becomes sufficiently small compared to the level of the fundamental component. The poles and residues of $h_1(t)$ be will identified using the pencil-of-functions approach (Reference 5). The pencil-of-functions approach integrates the input to the linear system and resulting output N times over the real-time interval (0,T).

It has been shown (Reference 5) that poles of the linear system satisfy the polynomial equation

$$\sum_{i=0}^{N} \lambda^{N-i} \left([G_{N2+1}]_{i+1,i+1} \right)^{1/2} = 0 \tag{8}$$

where G_{2N+1} is the Gram determinant shown in equation (9) below:

$$G_{2N+1} = \begin{vmatrix} \langle \bar{y}_1, \bar{y}_1 \rangle & \langle \bar{y}_1, \bar{y}_2 \rangle & \cdots & \langle \bar{y}_1, \bar{y}_{N+1} \rangle & \langle \bar{y}_1, x_2 \rangle & \cdots & \langle \bar{y}_1, x_{N+1} \rangle \\ \langle \bar{y}_2, \bar{y}_1 \rangle & \langle \bar{y}_2, \bar{y}_2 \rangle & \cdots & \langle y_2, \bar{y}_{N+1} \rangle & \langle \bar{y}_2, x_2 \rangle & \cdots & \langle \bar{y}_2, x_{N+1} \rangle \\ \vdots & & & & & & \\ \langle \bar{y}_N, \bar{y}_1 \rangle & \langle \bar{y}_N, \bar{y}_2 \rangle & \cdots & \langle \bar{y}_N, \bar{y}_{N+1} \rangle & \langle \bar{y}_N, x_2 \rangle & \cdots & \langle \bar{y}_N, x_{N+1} \rangle \\ \langle x_2, \bar{y}_1 \rangle & \langle x_2, \bar{y}_2 \rangle & \cdots & \langle x_2, \bar{y}_{NH} \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_{N+1} \rangle \\ \vdots & & & & & & \\ \langle x_{N+1}, \bar{y}_1 \rangle & \langle x_{N+1}, \bar{y}_2 \rangle & \cdots & \langle x_{N+1}, \bar{y}_{N+1} \rangle & \langle x_{N+1}, x_2 \rangle & \cdots & \langle x_{N+1}, x_{N+1} \rangle \end{vmatrix} \tag{9}$$

and where

$$x_{i+1}(t) = \begin{cases} \int_0^t x_i(\tau)\,d\tau & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{10}$$

$i = 1, \ldots, N$

$$\bar{y}_{i+1}(t) = \begin{cases} \int_0^t \bar{y}_i(\tau)d\tau & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

Further, the residues R_i of the poles λ_i satisfy the equation

$$R = C^{-1}Y \tag{12}$$

where

$$R = \text{residue matrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_N \end{bmatrix} \tag{13}$$

$$Y = \text{output matrix} = \begin{bmatrix} \bar{y}_2(T) \\ \bar{y}_3(T) \\ \bar{y}_4(T) \\ \vdots \\ \bar{y}_{N+1}(T) \end{bmatrix} \tag{14}$$

C = N x N matrix whose i,j^{th} element is defined by

$$C_{ij} = \frac{P_j(T)}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{x_{m+1}(T)}{(\lambda_j)^{i+1-m}} \tag{15}$$

where

$$P_j(T) = \int_0^T e^{\lambda_j(T-\tau)} x(\tau)d\tau \tag{16}$$

References 1 and 3 describe how this processing can be used to to determine N.

2. Identification of the Second Order Impulse Response, $h_2(t_1,t_2)$

The second step of the identification procedure is to identify the unknown parameters of $h_2(t_1, t_2)$. With $h_2(t_1,t_2)$ given by:

$$h_2(t_1,t_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_1+a_{k_2}t_2} U(t_2 \text{[illegible]}$$

$$+ \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_2+a_{k_2}t_1} U(t_1 - t_2) \qquad (17)$$

the only unknown parameters are the $A_{k_1k_2}$ quantities since M, N, A_{k1} and A_{k2} are known from identification of $h_1(t)$. A procedure for determining the $A_{k_1k_2}$ using the pencil-of-functions method is described in this section.

The identification procedure utilizes the response of the weakly nonlinear system to a sum of L decaying exponentials as described by:

$$x(t) = \begin{cases} \sum_{i=1}^{L} e^{-\alpha_i t} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases} \qquad (18)$$

where Re $\{\alpha_i\} > 0$. The second-order portion of the response to x(t) is given by

$$Y_2(s) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} A_{k_1 k_2}$$

$$\cdot \left[\frac{\alpha_i + \alpha_j + 2a_{k_1}}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_1})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \left(\frac{1}{s - (a_{k_1} + a_{k_2})} \right) \right.$$

$$- \frac{1}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_2})} \left(\frac{1}{s + (\alpha_j - a_{k_2})} \right)$$

$$- \frac{1}{(\alpha_i + a_{k_1})(\alpha_j + a_{k_2})} \left(\frac{1}{s + (\alpha_i - a_{k_2})} \right)$$

$$\left. + \frac{\alpha_i + \alpha_j + 2a_{k_2}}{(\alpha_j + a_{k_2})(\alpha_i + a_{k_2})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \left(\frac{1}{s + \alpha_i + \alpha_j} \right) \right] \qquad (19)$$

where

$$\alpha_j \neq a_{k_1} \qquad \text{for } j = 1,\ldots,L;\ k_1 = 1,\ldots,M$$

$$\alpha_i + \alpha_j + a_{k_1} + a_{k_2} \neq 0 \text{ for } i,j = 1,\ldots,L;\ k_1 = 1,\ldots,M;$$

$$k_2 = 1,\ldots,N$$

$$a_{k_1} \neq a_{k_2} \qquad \text{for } k_2 = 1,\ldots,N;\ k_1 = N+1,\ldots,M. \qquad (20)$$

The expression in equation (19) is the Laplace transform of a sum of exponential time functions. This sum can be interpreted as the impulse response of an equivalent linear system as indicated in Figure 1. In other words, the second-order response $y_2(t)$ can be visualized as though it were generated by an equivalent linear system. However, the equivalence is valid only if the equivalent linear system is considered to be excited by an impulse. It follows that the problem of identifying $h_2(t_1,t_2)$ has been reduced to the simpler problem of identifying a linear system and the pencil-of-functions technique can be used again.

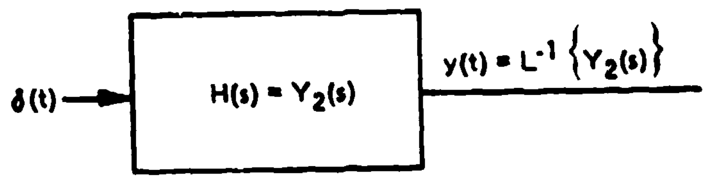


Figure 1. Equivalent Linear System with Transfer Function $Y_2(s)$

The system is excited by an input amplitude such that the output is described by linear and second-order terms, $y_1(t)$ and $y_2(t)$. The identification process will operate on the signal $y_2(t)$. For this purpose, the second-order portion of the response, $y_2(t)$, is isolated from the total response. $y_2(t)$ is obtained by subtracting from the total response, the corresponding linear response $y_1(t)$, which is known because $h_1(t)$ has been identified. It is shown in Reference 2 and in Section III.F of this report that the second-order response $y_2(t)$ need not be isolated from the total response for the identification procedure to work. However, isolation of $y_2(t)$ from the total response eases the mathematical presentation and is assumed at this point.

Once $y_2(t)$ is isolated from the total response, the coefficients $A_{k_1k_2}$ are then evaluated by applying the pencil-of-functions method to $y_2(t)$, treating it as though it were the impulse response of a linear system. This latter step is now discussed in detail.

From equation (19), the poles of $Y_2(s)$ are given by

$$s = a_{k_1} + a_{k_2},\ k_1 = 1,\ldots,M;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i + a_{k_2},\ i = 1,\ldots,L;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i - \alpha_j,\ i,j = 1,\ldots,L. \qquad (21)$$

First, consider poles of the form $s = a_{k_1} + a_{k_2} = 2\lambda_\ell$; $\ell = 1,\ldots,N$. The terms in $Y_2(s)$ corresponding to the pole at $2\lambda_\ell$ are given by

$$Y_{2\ell\ell}(s) = \sum_{i=1}^{L} \sum_{j=1}^{L} A_{\ell\ell} \frac{1}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)} \left(\frac{1}{s - 2\lambda_\ell}\right).$$

If the residue of the pole at $2\lambda_\ell$, as evaluated using the pencil-of-functions method, is $\beta_{\ell\ell}$, it follows that

$$A_{\ell\ell} = \beta_{\ell\ell} \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{1}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)} \qquad \ell = 1,\ldots,N. \tag{23}$$

This procedure results in identification of N of the coefficients.

Consider next poles of the form $s = a_{k_1} + a_{k_2} = \lambda_\ell + \lambda_m$ where $\ell \neq m$ and $\ell, m = 1,\ldots,N$. Since $A_{\ell m} = A_{m\ell}$ for $\ell, m \leq N$, the terms in $Y_2(s)$ corresponding to the pole at $\lambda_\ell + \lambda_m$ are given by

$$Y_{2\ell m}(s) = \sum_{i=1}^{L} \sum_{j=1}^{L} A_{\ell m}$$

$$\cdot\left[\frac{\alpha_i + \alpha_j + 2\lambda_\ell}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)} + \frac{\alpha_i + \alpha_j + 2\lambda_m}{(\alpha_j + \lambda_m)(\alpha_i + \lambda_m)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)}\left(\frac{1}{s - \lambda_\ell - \lambda_m}\right)\right]. \tag{24}$$

If the residue of the pole at $\lambda_\ell + \lambda_m$, as evaluated using the pencil-of-functions method, is $\beta_{\ell m}$, it follows that

$$A_{\ell m} = \beta_{\ell m} \left\{\sum_{i=1}^{L} \sum_{j=1}^{L} \left[\frac{\alpha_i + \alpha_j + 2\lambda_\ell}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)} + \frac{\alpha_i + \alpha_j + 2\lambda_m}{(\alpha_j + \lambda_m)(\alpha_i + \lambda_m)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)}\right]\right\}^{-1} \qquad \begin{array}{l}\ell, m = 1,\ldots,N \\ \ell \neq m,\ \ell < m.\end{array} \tag{25}$$

This procedure results in identification of $N(N - 1)/2$ of the coefficients.

The remaining unknown N^2 coefficients cannot be evaluated directly, as was done in equations (23) and(25), because the residues of the other poles in $Y_2(s)$ involve linear combinations of more than one unknown coefficient. However, if the number of exponential input signals, L, is set equal to N, N^2 linearly independent equations involving the N^2 unknown $A_{k_1k_2}$ coefficients can be obtained by considering the poles of $Y_2(s)$ of the form $s = -\alpha_j + \lambda_i$, $i = 1,\ldots,N$; $j = 1,\ldots,N$, in a manner similar to the above analysis. This fact is proven in Reference 3. Solution of the N^2 equations completes the identification process.

SECTION III

SYSTEM COMPLEXITY STUDY

A. COMPUTATIONAL COMPLEXITY CONSIDERATIONS

The analyses of Section II and Reference 3 demonstrated that the theoretical derivation of the identification technique was restricted to a class of nonlinear systems described by the impulse response of the form

$$h_1(t) = \begin{cases} \sum_{i=1}^{N} R_i \, e^{\lambda_i t} & , \; t \geq 0 \\ 0 & , \; t < 0 \end{cases} \tag{26}$$

Although there appears to be no theoretical limitation preventing the application of the identification technique to systems with multiple-order poles, the analysis has not been done to support this conclusion. The systems modeled by equation (26) represent a broad class of nonlinear systems to which the technique can be applied. Practical limitations of the technique will restrict the class of systems to a subset of those represented by equation (26). These practical limitations arise primarily from the computational requirements of the identification technique processing scheme. These limitations constrain the maximum value of N, which restricts application of the technique to systems whose linear incremental model has N poles or less.

This section investigates these computational limitations, attempts to establish a maximum value for N, and presents selected techniques to alleviate these computational problems.

1. Computational Complexity Limitations

The numerical computation requirements of the identification technique are summarized below. They are:

(1) N numerical integrations of input and output (N is the order of the system).

(2) Formation of 2N + 1 inner product entries for the Gram matrix.

(3) Evaluation of determinants of $N(2N+1)$ matrices.

(4) Solution of a N^{th} order polynomial equation.

(5) Evaluation of the N^2 C_{ij} entries of the C matrix in the residue equation.

(6) Inversion of an N dimension C matrix.

(7) Solution of the $A_{k_1k_2}$ quantities (second-order system).

The numerical accuracy requirements for these computations were investigated in Part I of this study (Reference 1). The numerical accuracy required for satisfactory performance of the identification technique increases significantly with increasing N. The severest computation requirements are imposed by: (1) the formation and inversion of the C matrix used in the residue equation $R = C^{-1} Y$; and (2) numerical integration and formation of the inner products for the appropriate Gram matrix. These two areas are addressed below.

2. Residue Equation Computational Requirements

The residue equation for the identification technique is given by

$$R = C^{-1} Y \tag{27}$$

The C matrix has dimension N' where N' is the order of the system being identified. The C matrix entries are given by

$$C_{ij} = \frac{e^{\lambda_j T}}{\lambda_j i} \int_0^T e^{-\lambda_j \tau} x(\tau)\, d\tau - \sum_{m=1}^{i} \frac{x_{m+1}(T)}{\lambda_j^{(i+1-m)}} \tag{28}$$

where the λ_j, $j = 1,\ldots,N'$ are the poles of the system being identified.

For the identification technique, x(t) is of the form

$$x(t) = e^{-\alpha_i t}, \quad t \geq 0 \quad \text{(linear system identification)} \tag{29}$$

or

$$x(t) = \delta(t) \quad \text{(secondary system identification)} \tag{30}$$

These x(t) expressions reduce C_{ij} to:

(1) for $x(t) = e^{-\alpha_k t}$ $(\alpha_k > 0)$

$$C_{ij} = \frac{1}{(\lambda_j + \alpha_k)\,\lambda_j^i}\left[e^{\lambda_j T} - e^{-\alpha_k T}\right] - \sum_{m=1}^{i} \frac{(e^{-\alpha_i T} - 1)}{(-\alpha_i)^m \lambda_j^{(i + 1 - m)}} \tag{31}$$

(2) for $x(t) = \delta(t)$

$$C_{ij} = \frac{e^{\lambda_j T}}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{(T)^{m - 1}}{(m - 1)!(\lambda_j^{i + 1 - m})} \tag{32}$$

It has been noted in previous work (References 2, 4) that the C matrix tends to be ill-conditioned, which hampers its computational inversion. Two significant problems complicate the inversion problem: (1) the dynamic range of the C_{ij}; and (2) the near singularity of the C matrix when two of the N' poles are nearly equal. The singularity problem is addressed in Sections III.B and C while the dynamic range problem is addressed below.

The computational requirements of inverting the C matrix are basically determined by its dimension. The dimension of the C matrix is determined by the number of poles of the system being identified. Consider a nonlinear system whose linear transfer function has N poles. The dimension of the C matrix is then N x N. Identification of the second-order transfer function involves the identification of the residues of the second-order response, $Y_2(s)$. The number of poles of $Y_2(s)$ is

$$N' = N\left(\frac{N}{2} + \frac{3}{2} + L\right) + \frac{L\,(L + 1)}{2} \tag{33}$$

where L is the number of exponential signals composing the input. For the general case where L = N,

$$N' = 2N\,(N + 1) \tag{34}$$

Figure 2 plots N' as a function of N. It is noted that, for a 10-pole linear system, the calculation of the residues of $Y_2(s)$ involves inversion of a 240 x 240 matrix. The dimension of the C matrix for second-order identification grows rapidly with the number of linear system poles.

The dynamic range of the C_{ij} entries becomes significant as the dimension of the C matrix increases since the C_{ij} entries are inversely proportional to λ_j^{i}, for $i = 1,\ldots,N'$. This is demonstrated below.

For identification of the residues of $Y_2(s)$, the input x(t) is given by

$$x(t) = R_a \; \delta(t) \tag{35}$$

where $\delta(t)$ is the unit impulse. The C_{ij} entries for this input are given by

$$C_{ij} = R_a \left[\frac{e^{\lambda_j T}}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{T^{m-1}}{(\lambda_j)^{i+1-m} \, (m-1)!} \right] \tag{36}$$

Suppose the system response of interest, $Y_2(s)$, has a maximum pole/minimum pole ratio = 10. Further, assume that

$$-10 \leq \lambda_j T \leq -1$$

or that $T = -1/\lambda_{min}$.

For the minimum pole $\lambda_{j\ min}$

$$C_{ij} = R_a \left[\frac{e^{-1}}{(-1)^i} - \sum_{m=1}^{i} \frac{1}{(-1)^i \, (m-1)!} \right] \tag{37}$$

For different values of i, C_{ij} is given by

i	C_{ij}
1	$1.37\ R_a$
2	$-1.63\ R_a$
⋮	
10	$-0.35\ R_a$

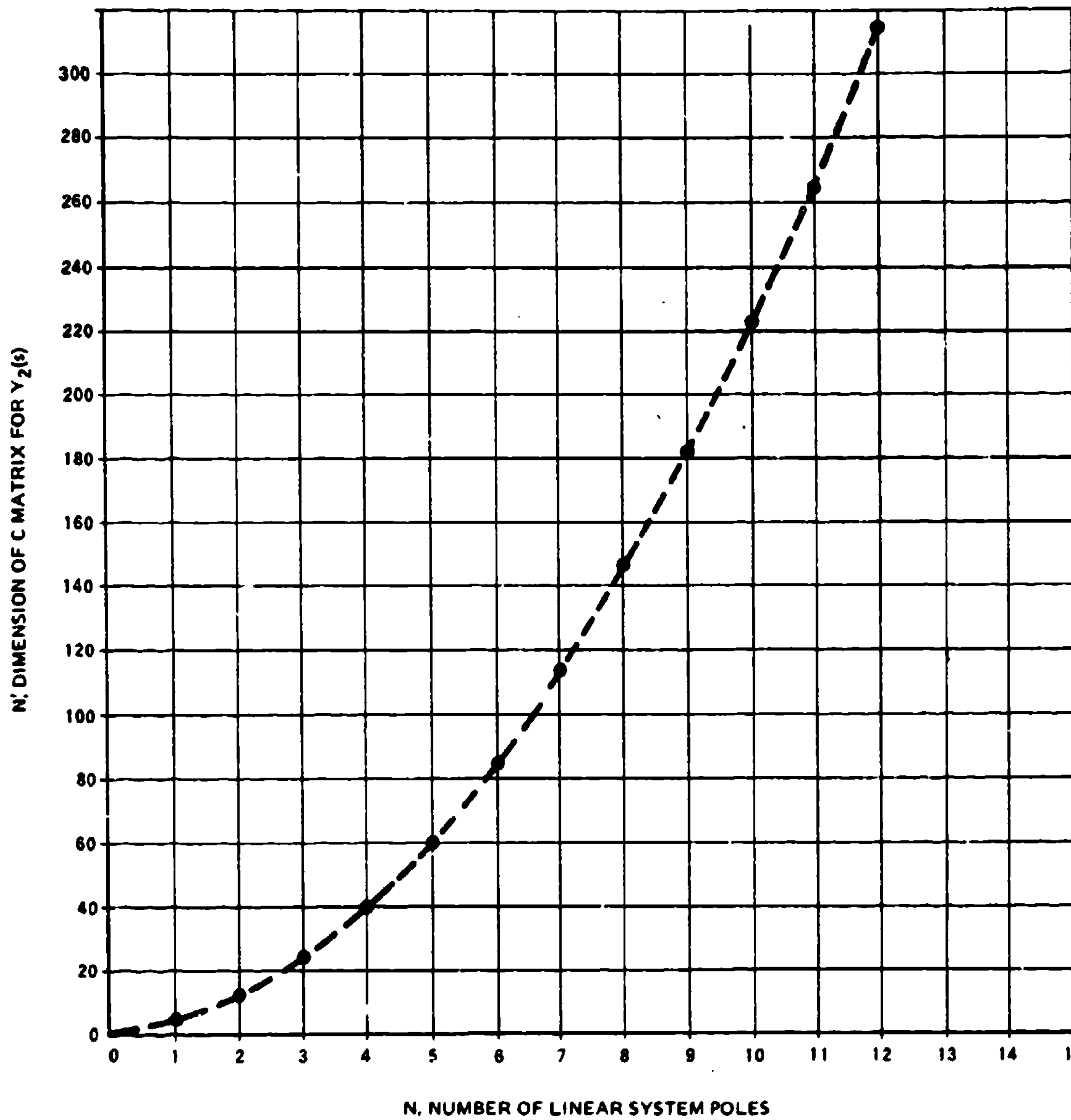


Figure 2. Number of Poles of $Y_2(s)$ as a Function of the Number of Linear System Poles

For the maximum pole $\lambda_{j\ max}$

$$C_{ij} = R_a \left[\frac{e^{-10}}{(-10)^i} - \sum_{m=1}^{i} \frac{(-1)^{m-1}}{(-10)^{i+1-m}(m-1)!}\right] \tag{38}$$

Again, for different values of i, C_{ij} is given by

i	C_{ij}
1	-0.1
2	0.09
⋮	
10	10^{-8}
⋮	
10	10^{-100}

The dynamic range of the C_{ij} entries for a C matrix of dimension 100 is approximately 10^{100}. For N' greater than 100, the dynamic range is even greater. This dynamic range can cause significant difficulty when the inverse of the C matrix is evaluated. For a typical general-purpose computer, the maximum computable dynamic range is 10^{76} (10^{-38} to 10^{38}). Furthermore, matrix inversion involves multiplication and division of pairs of matrix entries. The resultant product or division must be in the allowable dynamic range, which implies that the individual matrix entries must be well within the dynamic range.

For a system with two poles of its linear transfer function with a ratio of 10, the computer limitation constrains the class of systems to those with $N \leq 4$. This problem can be alleviated somewhat with scaling but not significantly.

A method of reducing the dimension of the C matrix must be found to ease the computational limitations. This requires that the number of poles $Y_2(s)$ be reduced.

A significant reduction in the number of poles of $Y_2(s)$ results if L = 1 instead of L = N, where L is the number of exponentials used in the input function. For L = 1, the number of poles of $Y_2(s)$ is given by

$$N' = N\left(\frac{N}{2} + \frac{5}{2}\right) + 1 \tag{39}$$

This is plotted in Figure 3 along with the plot for L = N. It is noted that the number of poles of $Y_2(s)$ is significantly less for L = 1.

It is recalled that L = N was required in order to generate a linearly independent set of equations from the residue of the poles at $s = -\alpha_i + a_{k_2}$, $i = 1,\ldots,L$; $k_2 = 1,\ldots,N$. These equations can also be obtained by using L = 1 and exciting the system with N individual inputs and recording the response to each input. This obviously complicates the identification procedure but does reduce the magnitude of the matrix inversion problem.

If the approach is adopted (L = 1), then the identification procedure is modified as follows. The system is excited by the input $x(t) = e^{-\alpha_1 t}$. The resultant output is of the form

$$Y_2(s) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} \left[\frac{2}{(\alpha_1 + a_{k_1})(2\alpha_1 + a_{k_1} + a_{k_2})} \cdot \left(\frac{1}{s - (a_{k_1} + a_{k_2})}\right) - \frac{2}{(\alpha_1 + a_{k_1})(\alpha_1 + a_{k_2})}\left(\frac{1}{s + (\alpha_1 + a_{k_2})}\right) + \frac{2}{(\alpha_1 + a_{k_2})(2\alpha_1 + a_{k_1} + a_{k_2})}\left(\frac{1}{s + 2\alpha_1}\right)\right] \tag{40}$$

The number of poles in $Y_2(s)$ above is $(N/2)(N + 5) + 1$. The $A_{k_1k_2}$ quantities of the form A_{ii}, $i = 1,\ldots,N$ are identified directly from the residues of the poles at $s = 2a_i$. The $A_{k_1k_2}$ quantities of the form, $A_{\ell m}$, $\ell \neq m$, $\ell, m = 1,\ldots,N$ are identified directly from the residues of the poles at $s = a_\ell + a_m$. This procedure identifies $N + [(N - 1)/2] N$ $A_{k_1k_2}$ quantities.

The system is then excited by the input $x(t) = e^{-\alpha_2 t}$. The resultant $Y_2(s)$ has $(N/2)(N + 5) + 1$ poles but the contributions from $N + [(N - 1)N/2]$ poles are known and may be subtracted out from the second-order response. The resultant $Y_2(s)$ has $2N + 1$ poles. The resultant C matrix is reduced to order $2N + 1$.

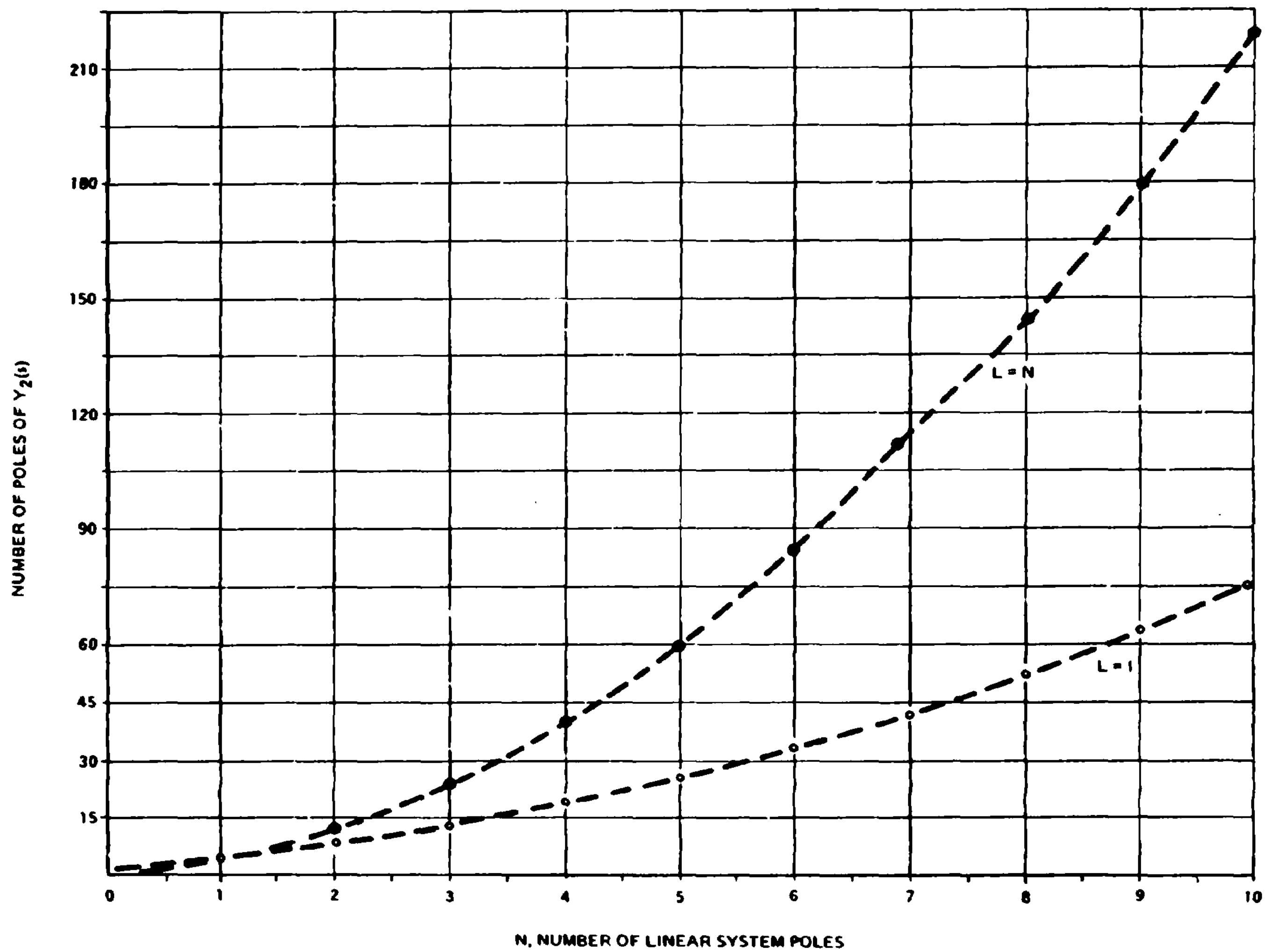


Figure 3. Number of Poles of $Y_2(s)$ as a Function of the Number of Linear System Poles and Exponential Inputs

This process is repeated for each input $x(t) = e^{-\alpha_i t}$, $i = 3, \ldots, N$. Each process results in a set of equations involving the residue at the poles $s = -\alpha_i + a_{k_2}$. The first application involves inverting an $(N/2)(N + 5) + 1$ matrix while the remaining $(N - 1)$ applications require inversion of a $2N + 1$ matrix.

The advantage of this approach is the reduction achieved in the number of poles of $Y_2(s)$. This eases the computational complexity of matrix inversion. The primary disadvantage of the technique is that the identification measurement process must be repeated N times in order to identify $h_2(t_1, t_2)$.

Another alternative approach to reducing the number of poles of $Y_2(s)$ without significantly increasing the measurement process is to initially excite the system with an input, $x(t) = e^{-\alpha_1 t}$. The residues of the poles at $s = \lambda_i + \lambda_j$, $i, j = 1, \ldots, N$, are evaluated to determine the appropriate $A_{k_1 k_2}$ quantities. The system is then excited with an input

$$x(t) = \sum_{i=1}^{N} e^{-\alpha_i t}$$

Since $N + [N(N - 1)/2]$ $A_{k_1 k_2}$ quantities have been identified above, the contributions of the associated poles can be subtracted from the total response. The resultant response has $(3N/2)(N + 1)$ poles. This is a reduction of 25 percent or $(N/2)(N + 1)$ from the original second-order response obtained with

$$x(t) = \sum_{i=1}^{N} e^{-\alpha_j t}.$$

Although the order of reduction is not as great as that achieved by applying the input $x(t) = e^{-\alpha_1 t}$ N times, the identification measurement process need be repeated only twice.

The basics of these approaches to the identification process are summarized in Table 1 for comparison purposes.

The best approach for the identification technique is dependent on the system under test. The number of linear system poles and the ratio of the maximum to minimum poles dictate the complexity of the matrix inversion problem and, in turn, determine which of the above approaches will maximize the performance of the identification technique. Therefore, selection of the best approach must wait until the linear portion of the system under test has been identified.

TABLE 1. COMPARISON OF APPROACHES DESIGNED TO ALLEVIATE THE COMPUTATIONAL COMPLEXITY OF THE IDENTIFICATION TECHNIQUE

Identification Procedure	Order of $Y_2(s)$	Matrix Inversion Requirements	Relative Advantages/ Disadvantages
Apply input $x(t) = \sum_{i=1}^{N} e^{-\alpha_i t}$	$2N\ (N + 1)$	Invert a $2N\ (N + 1)$ dimension matrix	Single measurement set/ dimension limited
Apply input $x(t) = e^{-\alpha_i t}$ N times (changing α_i)	$(\frac{N}{2})\ (N + 5) + 1$	Invert one $(N/2)(N + 5) + 1$ dimension matrix; Invert $N - 1$ different $2N + 1$ dimension matrices	Minimum dimension inversion/ N separate measurement sets
Apply input $x(t) = e^{-\alpha_i t}$ Then apply input $x(t) = \sum_{i=1}^{N} e^{-\alpha_i t}$	$(\frac{N}{2})\ (N + 5) + 1$ $(3\frac{N}{2})\ (N + 1)$	Invert one $(N/2)(N + 5) + 1$ dimension matrix; Invert one $(3N/2)(N + 1)$ dimension matrix	Reduced dimension inversion/ Two separate measurement sets

Another potential approach to reducing the computational complexity is the frequency range separation technique of Jain and Osman (Reference 2). This approach divides the frequency extent of a given system into a set of frequency ranges, e.g., low frequency region, middle to high frequency transition region, and high frequency region. The system is excited by an input signal that is approximately matched to the frequency region of interest and the integration time is selected consistent with this frequency range. The identified transfer function is then a representation of the system transfer function in the specified frequency region.

This approach assumes some a priori knowledge of the poles of the system transfer function in order to permit frequency region separation and determination of the number of poles of the system transfer function in each region. Since, for the nonlinear system identification technique of interest in this study, the order and pole locations of the second-order response are known, the above requirement is satisfied. The input function for the nonlinear system identification technique,

$$x(t) = \sum_{i=1}^{N} e^{-\alpha_i t},$$

could be divided into the frequency regions of interest and applied separately for each frequency region. For example, if $N = 6$, and $Y_2(s)$ is divided into three frequency ranges, then the identification procedure is conducted as follows. Three sets of measurements are taken, each with input

$$x(t) = \sum_{i=1}^{2} e^{-\alpha_i t}$$

where the α_i are selected consistent with the frequency region of interest. The three sets of measurements are then collectively used to solve for the $A_{k_1 k_2}$ quantities in the normal manner.

The achievable reduction in computational complexity using this approach is dependent on the characteristics of the system under test. If the poles of $Y_2(s)$ are distributed uniformly in frequency, then the number of poles of $Y_2(s)$ in each of three frequency regions is $N'/3$ and a one-third reduction in the size of the matrices to be inverted has been realized. It is necessary to point out that the price of this reduction is the need to perform the identification procedure three times instead of once.

3. Numerical Integration Techniques

The results of the implementation feasibility portion (Part I) of this study indicated that significant accuracy was required of the numerical integration technique to achieve satisfactory technique performance. The results indicated that the primary source of inaccuracy in the integration was the quantization error introduced by the A/D converter. This will be the case independent of the numerical integration technique used. However, the results of Part I of the study also indicated that the Simpson's rule of numerical integration technique introduced numerical inaccuracy for higher-order systems. The reason for this inaccuaracy was the fact that Simpson's rule reduces the number of samples on each successive integration. Simpson's rule of integration (Reference 2) is given by

$$\int_a^b y(t)\,dt = \frac{(b - a)}{6\,n}\; y(0) + 4y(\Delta T) + 2y(2\Delta T) + 4y(3\Delta T) + \ldots + 2y((2n - 2)\Delta T) + 4y((2n - 1)\Delta T) + y(2n\Delta T) \tag{41}$$

where

$\Delta T = (b - a)/2n$ = time between samples

$2n$ = number of subintervals between data points.

Each integration using the Simpson's rule integration technique results in a reduction of the number of samples that can be used for the next integration. This is illustrated below.

Consider the output samples $y_1(0)$, $y_1(T)$, $y_1(2T), \ldots,$ $y_1(2nT)$, where nT is the n^{th} sample and T is the sampling interval. The integral of $y_1(t)$,$y_2(t)$, as obtained using Simpson's rule, is given by the samples

$$y_2(0),\; y_2(2T),\; y_2(4T), \ldots, y_2(2nT).$$

It is noted that there are only nT samples of $y_2(t)$ whereas there were $2nT$ samples of $y_1(t)$. As this output is successively integrated, the time distance between samples increases and the numerical accuracy of the integration technique decreases. This will have an adverse effect on the performance of the identification technique, especially for higher-order systems.

A way of alleviating this decreasing-number-of-samples problem is to interpolate between samples output from Simpson's rule of integration. For instance, in the above example, Simpson's rule produced an output at t = 0 and t = 2T, namely, $y_2(0)$ and $y_2(2T)$. Interpolating linearly between samples yields

$$y_2(T) = \frac{y_2(2T) + y_2(0)}{2} \tag{42}$$

To determine the impact of this procedure on the performance of the identification technique, this procedure was added to the computer simulation of the identification technique. The computer simulation of the identification technique was discussed in detail in Part I of this final report (Reference 1). The simulation was run for two systems, a two-pole and a four-pole system. The results are presented in Tables 2 and 3. The performance of the direct application of Simpson's rule is included for comparison in Tables 2 and 3.

The results of Table 2 indicate that, for a two-pole system, the interpolation procedure slightly degrades the performance of the technique. The results of Table 3 indicate that, for a four-pole system, the interpolation scheme offers slightly improved performance for A/D converters with 20 bits or less of resolution. This improvement will continue to be evident as system order increases. These results suggest that the numerical integration technique be modified to include this interpolation method.

B. MATRIX INVERSION/SCALING TECHNIQUES

The primary computational problem of the nonlinear system identification technique is the matrix inversion involved in solving the residue equation.

The matrix to be inverted has entries given in general by

$$C_{ij} = \frac{P_j(T)}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{x_{m+1}(T)}{(\lambda_j)^{i+1-m}} \tag{43}$$

where

$$P_j(T) = \int_0^T e^{\lambda_j(T-\tau)}\, x(\tau)\, d\tau \tag{44}$$

For the second-order system identification, $x(t) = \delta(t)$, a unit impulse which reduces the matrix entries to

TABLE 2. IMPACT OF INTERPOLATION BETWEEN SAMPLES ON THE INTEGRATION FOR TWO POLE SYSTEM. INTEGRATION TIME = 9.6 μs, SAMPLING TIME = 4 ns

Actual System Poles (MHz)	Actual System Residues
0.011550998	2.8069192×10^{5}
10.616986	-2.7368441×10^{8}

Number of A/D Bits	Predicted System Poles (MHz)	Percentage Error	Predicted System Residues	Percentage Error	Normalized Mean Squared Error (NMSE)	NMSE for Unmodified Numerical Integration
No A/D	0.0115417365 10.65798117	-0.054 0.386	2.8061864×10^{5} $-2.74727291 \times 10^{8}$	-0.026 0.381	0.253×10^{-7}	0.411×10^{-8}
16	0.011541585 10.657335	-0.081 0.38	$2.806345709 \times 10^{5}$ $-2.74728918 \times 10^{8}$	-0.0204 0.382	0.245×10^{-7}	0.147×10^{-7}
14	0.011531168 10.654727	-0.17 0.355	$2.806234938 \times 10^{5}$ $-2.74690418 \times 10^{8}$	-0.0243 0.367	0.15×10^{-6}	0.174×10^{-5}
12	0.0115420827 10.6474118	0.30 0.395	2.81535417×10^{5} $-2.747664193 \times 10^{8}$	-0.077 0.286	0.103×10^{-4}	0.185×10^{-5}
10	0.0100396924 10.3686832	-13.1 -2.33	2.7079274×10^{5} -2.6821995×10^{8}	-3.526 1.99	0.637×10^{-3}	0.49×10^{-4}
8	0.0002729567 10.9527208	-97.6 3.16	-8.[illegible]6222 $\times 10^{3}$ -1.2301584×10^{8}	-103 -55	1.08	0.367

TABLE 3. IMPACT OF INTERPOLATION BETWEEN SAMPLES ON INTEGRATION FOR FOUR POLE SYSTEM. INTEGRATION TIME = 4.8 μs, SAMPLING TIME = 1.5 ns

Actual System Poles (MHz)	Actual System Residues
0.011550998	2.8069192×10^{5}
0.510	-1.20×10^{7}
0.82	1.51×10^{7}
6.5	-1.61×10^{8}

Number of A/D Bits	Predicted System Poles (MHz)	Percentage Error	Predicted System Residues	Percentage Error	Normalized Mean Squared Error (NMSE)	NMSE for Unmodified Numerical Integration
No A/D	0.011543643	-0.0036	2.80343709×10^{5}	-0.124	0.296×10^{-4}	0.103×10^{-4}
	0.5105753969	0.113	-1.1798384×10^{7}	-1.68		
	0.817877528	-0.259	1.44687812×10^{7}	-4.18		
	6.5077115	0.118	$-1.59170892 \times 10^{8}$	-1.13		
24	0.011543124	-0.0613	$2.805015188 \times 10^{5}$	-0.0678	0.659×10^{-5}	0.108×10^{-5}
	0.51057176	0.112	-1.1964009×10^{7}	-0.447		
	0.8178845017	-0.258	1.4830854×10^{7}	-1.78		
	6.507708137	0.118	$-1.60188023 \times 10^{8}$	-0.504		
20	0.0115420008	-0.0779	2.8266673×10^{5}	0.703	0.14×10^{-2}	0.2×10^{-2}
	0.510672573	0.132	$-1.40317097 \times 10^{7}$	16.93		
	0.817040863	-0.288	1.99001358×10^{7}	31.79		
	6.50786798	0.121	$-1.74376249 \times 10^{8}$	8.3		
16	0.0114817501	-0.6	3.215878×10^{5}	14.5	0.581	0.131×10^{1}
	0.51186235	0.365	-5.2221537×10^{7}	335.0		
	0.81538073	-0.563	1.1194778×10^{8}	641.0		
	6.508542926	0.131	$-4.306279876 \times 10^{8}$	167.0		

$$C_{ij} = \frac{e^{\lambda_j T}}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{T^{m-1}}{(m-1)!\left(\lambda_j^{\,i+1-m}\right)} \tag{45}$$

This expression can be rewritten as

$$C_{ij} = \frac{1}{\lambda_j^{\,i}} = \left[e^{\lambda_j T} - \sum_{m=1}^{i} \frac{(\lambda_j T)^{m-1}}{(m-1)!} \right] \tag{46}$$

But since

$$e^{\lambda_j T} = \sum_{m=1}^{\infty} \frac{(\lambda_j T)^{m-1}}{(m-1)!} \tag{47}$$

for λ_j real, the C_{ij} entries become

$$C_{ij} = \frac{1}{\lambda_j^{\,i}} \left[\sum_{m=i+1}^{\infty} \frac{(\lambda_j T)^{m-1}}{(m-1)!} \right] \tag{48}$$

or

$$C_{ij} = \frac{1}{\lambda_j^{\,i}} \sum_{m=i}^{\infty} \frac{(\lambda_j T)^{m}}{m!} \tag{49}$$

This expression serves to illustrate two basic problems with the numerical inversion of the C matrix. First, the dynamic range limitation of a digital computer limits the number of terms which can be used in a given summation. Also, it is clear that if two λ_j quantities used in the C_{ij} expression are nearly equal, the C_{ij} terms become nearly equal since the m! quantity tends to reduce the difference between the C_{ij} entries.

This numerical similarity between the C_{ij} entries causes the major problem incurred when attempting to invert the C matrix. Several approaches to alleviating this problem have been postulated (Ref. 2) and these provide some relief but do not eliminate the problem. The dynamic range of the matrix entries provides the major limitation and avoiding this problem requires intricate programming which essentially corresponds to scaling quantities after each operation. This is a long and involved process, and the design of such a program is beyond the scope of the present effort. However, it remains an important area to consider in future efforts because it may help to eliminate a major limitation of the application of the technique. The other techniques discussed in Reference 2 have been used to invert matrices which have the numerical structure of the C matrix and they offer some relief to the existing problem. These techniques are reviewed below.

First, it should be noted that standard matrix inversion techniques perform well when the C matrix is not nearly singular and/or large enough that the dynamic range limitations of the computer are exceeded. This was demonstrated in References 1 and 3 for two-, four- and eight-pole systems. The postulated approaches should be applied only when the standard techniques fail to perform satisfactorily.

The initial technique to aid in matrix inversion is called "adaptive scaling" (Reference 2). This technique depends on row and column scaling to alleviate the problem caused by matrix entries which differ widely in magnitude. If it is desired to invert a matrix C, the first step is to do a row and column scaling on C, transforming it to

$$C_o = PCQ \tag{50}$$

where P and Q are diagonal scaling matrices. The entries of the P and Q matrices can be chosen as follows. Consider the P matrix. The P_{ii} entry is computed as follows:

$$P_{ii} = \prod_{\substack{\text{qualifying} \\ \text{entries}}} \left\{ (C)_{ij} \right\}^{1/n_i} \tag{51}$$

where the qualifying entries of each row are determined from those C_{ij} entries where

$$\text{ABS } (C_{ij}) > \alpha_i * 10^{-m} \tag{52}$$

where $\alpha_i = \max_j \text{ ABS } (C_{ij})$ (largest entry of i^{th} row)

m is chosen by the user

n_i = number of C_{ij} entries that exceed $\alpha_i * 10^{-m}$ threshold

The scaled C matrix entries are then given by

$$C_{o_{ij}} = \frac{C_{ij}}{P_{ii}\, Q_{jj}} \tag{53}$$

The inverse matrix C^{-1} is found from

$$C_o^{-1} = Q^{-1}\, C\, P^{-1} \tag{54}$$

This involves evaluating three matrices as opposed to one but the two diagonal matrices are easily inverted.

In addition to the row and column scaling techniques, there are several perturbation methods proposed in Reference 2. These involve forming the matrix A from

$$A = C_o + \varepsilon D \tag{55}$$

where C_o is the scaled version of the original matrix and D is a diagonal matrix whose entries can be taken as those of the diagonal of A. The constant ε is chosen to be suitably small such that C is invertible. (Selection of ε is discussed in detail in Reference 2.) Then, the inverse of C is given by

$$\begin{aligned} C^{-1} &= (A - \varepsilon D)^{-1} \\ &= A^{-1} + \varepsilon A^{-1} D A^{-1} + \varepsilon^2 (C^{-1}D)^2 C^{-1} + \ldots \end{aligned} \tag{56}$$

C^{-1} is found using A^{-1}, ε and D, all of which are known. A^{-1} was found using standard matrix inversion techniques.

A problem with this approach is that C^{-1} is not found in closed form and then numerical accuracy becomes a question.

The techniques were exercised using the computer routine provided in Reference 2. These routines were obtained from Mr. Daniel Kenneally of Rome Air Development Center who received them from their originator, Dr. V. K. Jain (Reference 2).

The computer routines were exercised for the second-order response of a system with the linear transfer function given by

$$\begin{aligned} H(s) = & \frac{2.8069192 \times 10^5}{s + 0.011550998\ (2\pi) \times 10^6} \\ & - \frac{2.7368441 \times 10^8}{s + 10.616986\ (2\pi) \times 10^6} \end{aligned} \tag{57}$$

The input to the system was

$$x(t) = (e^{\alpha_1 t} + e^{\alpha_2 t})\ U(t) \tag{58}$$

where $\alpha_1 = 10^7$, $\alpha_2 = 1.75 \times 10^7$ rad/sec. The resultant second order response, $Y_2(s)$, has poles at

$$\lambda_1 = -0.011550998\ (2\pi) \times 10^6$$

$$\lambda_2 = -10.616986\ (2\pi) \times 10^6$$

$$\lambda_3 = 2\ \lambda_1 = -0.023101996\ (2\pi) \times 10^6$$

$$\lambda_4 = 2\ \lambda_2 = -21.233972\ (2\pi) \times 10^6$$

$$\lambda_5 = \lambda_1 + \lambda_2 = -10.628537\ (2\pi) \times 10^6$$

$$\lambda_6 = \alpha_1 + \lambda_1 = -1.603100429\ (2\pi) \times 10^6$$

$$\lambda_7 = \alpha_1 + \lambda_2 = -12.20853543\ (2\pi) \times 10^6$$

$$\lambda_8 = \alpha_2 + \lambda_1 = -2.796762502\ (2\pi) \times 10^6$$

$$\lambda_9 = \alpha_2 + \lambda_2 = -13.4021975\ (2\pi) \times 10^6$$

$$\lambda_{10} = 2\alpha_1 = -3.183098862\ (2\pi) \times 10^6$$

$$\lambda_{11} = 2\alpha_2 = -5.570423008\ (2\pi) \times 10^6$$

$$\lambda_{12} = \alpha_1 + \alpha_2 = -4.376760935\ (2\pi) \times 10^6 \qquad (59)$$

The C matrix entries are given by

$$C_{ij} = \frac{e^{\lambda_j T}}{\lambda_j^{\,i}} - \sum_{m=1}^{i} \frac{T^{m-1}}{(m-1)!\ \lambda^{i+1-m}} \qquad i,j = 1,\ldots,12 \qquad (60)$$

T was set to 600×10^{-9} second for this example.

The computer routines were exercised by varying the dimension of the C matrix from 8 to 12. In each case, the poles used were λ_j, $j = 1,\ldots,$matrix dimension. The results are given in Table 4. The computer routines evaluate the accuracy of the inverse as follows. The auxiliary matrix B_o is formed as

$$B_o = CC^{-1} - I \qquad (61)$$

where C is the original matrix. The RMSE is defined as

$$\text{RMSE} = \sum_{i=1}^{N} \sum_{j=1}^{N} (B_{o_{ij}})^2 \tag{62}$$

which is the sum of the squares of all the entries of B_o.

Note that $B_o = 0$ if C^{-1} is the exact inverse of C. Also included in Table 4 is the numerically evaluated determinant of the C matrix.

TABLE 4. MATRIX INVERSION RESULTS

C Matrix Dimension	Root Mean Squared Error	Determinant of C Matrix
8	0.149×10^{-7}	0.346×10^{-10}
9	0.383×10^{-5}	0.124×10^{-14}
10	0.44×10^{-3}	0.999×10^{-17}
11	0.21	0.222×10^{-22}
12	10^{10}	0

The results of Table 4 illustrate what happens as the matrix dimension increases. The two poles, λ_2 and λ_5, are nearly equal which causes the C matrix to be nearly singular. This condition becomes critical as the dimension increases above 11. The results of Reference [2] indicate that a similar matrix of dimension 12 was inverted. It appears that these computer results were generated on a computer whose dynamic range exceeds the 10^{38} to 10^{-38} capability of the computer used in this analysis.

Further it should be noted from the work of Reference [4] that standard matrix inversion techniques were able to invert this matrix when the dimension was 8 or less.

These techniques offer some potential relief from the matrix inversion problem encountered when using the identification technique. The dynamic range of the computer used to invert the matrix, however, remains the dominant limitation. A scaling of individual operations is perhaps a way of alleviating this problem but this is an extensive process. It appears at this point that the most viable way of increasing the applicability of the identification technique is to find methods of reducing the order of the second order system response, $Y_2(s)$. The primary focus of this effort, therefore, has been to examine ways of reducing the order of $Y_2(s)$ without restricting application of the identification technique. These techniques are evaluated in the following sections.

C. POLE APPROXIMATION APPROACH TO REDUCING ORDER OF $Y_2(s)$

The application of the identification technique to practical systems increases in difficulty as the ratio of the highest break frequency to the lowest break frequency of the linear system increases. The reason for this is that the poles of the second-order response include poles of the form

$$\lambda_\ell + \lambda_k \text{ and } \lambda_k,\ \ell,\ k = 1,\ \ldots,\ N$$

where the λ_i, $i = 1, \ldots, N$ are poles of the linear portion of the system.

If, for the system of interest,

$$\lambda_\ell + \lambda_k \approx \lambda_k$$

for some ℓ, k combination, it becomes extremely difficult to solve the residue equation

$$R = C^{-1}Y \tag{63}$$

where C is a matrix with entries of the form

$$C_{ij} = \frac{e^{a_j T}}{a_j^{\,i}} - \sum_{n=1}^{i} \frac{x_{m+1}(T)}{(a_j)^{i+1-m}} \tag{64}$$

where $a_j = \lambda_\ell + \lambda_k$, λ_ℓ, etc.

If two of the a_j are approximately equal, the C matrix is nearly singular and is extremely difficult to invert using standard matrix inversion techniques.

To expand the applicability of the identification technique, it is necessary to find a method of alleviating the computational problem discussed above. The approach to be taken in this section is to use the approximation

$$\lambda_i + \lambda_j = \lambda_j \quad \text{if } \lambda_i + \lambda_j \approx \lambda_j.$$

The question then is to determine the impact of this approximation on the identification procedure.

The first step is to derive the functional form of the second-order response, $Y_2(s)$, for this approximation. The approximation must be treated carefully to insure the correct expression for $Y_2(s)$ is obtained. This is demonstrated below.

Assume that we have a system output given by

$$Y(s) = \frac{A_1}{s + \lambda_1} + \frac{A_2}{s + \lambda_2} \tag{65}$$

The corresponding time function is given by

$$y(t) = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t} \tag{66}$$

If $\lambda_2 = \lambda_1 + \varepsilon$, then

$$y(t) = (A_1 + A_2\, e^{-\varepsilon t})\, e^{-\lambda_1 t} \tag{67}$$

and for $\varepsilon \to 0$,

$$y(t) \simeq (A_1 + A_2)\, e^{-\lambda_1 t} \tag{68}$$

and

$$Y(s) \simeq \frac{(A_1 + A_2)}{s + \lambda_1} \tag{69}$$

Therefore, $Y(s)$ is approximated by a single-pole system.

However, assume that the system output is of the form

$$Y'(s) = \frac{B}{(s + \lambda_1)(s + \lambda_2)} = \frac{\frac{B}{\lambda_2 - \lambda_1}}{s + \lambda_1} + \frac{\frac{B}{\lambda_1 - \lambda_2}}{s + \lambda_2} \tag{70}$$

The corresponding time response is given by

$$y'(t) = \frac{B}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \tag{71}$$

If $\lambda_2 = \lambda_1 + \epsilon$, then, as $\epsilon \to 0$

$$Y'(s) = \frac{B}{(s + \lambda_1)^2} \tag{72}$$

and

$$y'(t) = Bt\, e^{-\lambda_1 t} \tag{73}$$

In this case, $Y(s)$ is approximated by a double-pole system.

The approximation for $\lambda_i + \lambda_j = \lambda_i$ in $Y_2(s)$ must be made before the expression for $Y_2(s)$ is expanded into partial fractions. The second-order response, $Y_2(s_1, s_2)$, is given by

$$Y_2(s_1,s_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} A_{k_1k_2}$$

$$\cdot \left\{ \left[\frac{s_1 + s_2 - 2a_{k_2}}{(s_1 + s_2 - a_{k_1} - a_{k_2})(s_2 - a_{k_2})(s_1 - a_{k_2})} \right] \left(\frac{1}{s_1 + \alpha_i} \right) \left(\frac{1}{s_2 + \alpha_j} \right) \right\} \tag{74}$$

Expansion and simplification of $Y_2(s_1, s_2)$ yields

$$Y_2(s_1,s_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} \left(\frac{A_{k_1k_2}}{(\alpha_i + a_{k_2})(\alpha_j + a_{k_2})} \right)$$

$$\cdot \left[\left(\frac{s_1 + s_2 - 2a_{k_2}}{(s_1 + s_2 - a_{k_1} - a_{k_2})} \right) \right] \left[\left(\frac{1}{(s_1 - a_{k_2})(s_2 - a_{k_2})} \right) \right.$$

$$- \left(\frac{1}{(s_1 - a_{k_2})(s_2 + \alpha_j)} \right) - \left(\frac{1}{(s_1 + \alpha_i)(s_2 - a_{k_2})} \right)$$

$$\left. + \left(\frac{1}{(s_1 + \alpha_i)(s_2 + \alpha_j)} \right) \right] \tag{75}$$

Association of variables yields

$$Y_2(s) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{A_{k_1k_2}}{(\alpha_i + a_{k_2})(\alpha_j + a_{k_2})} \left[\frac{s - 2a_{k_2}}{s - (a_{k_1} + a_{k_2})}\right]$$

$$\cdot \left[\frac{1}{s - 2a_{k_2}} - \frac{1}{s + (\alpha_j - a_{k_2})} - \frac{1}{s + (\alpha_i - a_{k_2})} + \frac{1}{s + (\alpha_i + \alpha_j)}\right] \tag{76}$$

Equation (76) involves products of transforms. Specifically,

$$Y_2(s) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{A_{k_1k_2}}{(\alpha_i + a_{k_2})(\alpha_j + a_{k_2})} \left\{ \frac{1}{s - (a_{k_1} + a_{k_2})} \right.$$

$$- \frac{s - 2a_{k_2}}{\left[s - (a_{k_1} + a_{k_2})\right]\left[s + (\alpha_j - a_{k_2})\right]} - \frac{s - 2a_{k_2}}{\left[s - (a_{k_1} + a_{k_2})\right]\left[s + (\alpha_i - a_{k_2})\right]}$$

$$\left. + \frac{s - 2a_{k_2}}{\left[s - (a_{k_1} + a_{k_2})\right]\left[s + (\alpha_i + \alpha_j)\right]} \right\} \tag{77}$$

Since $\alpha_i \neq a_{k_1}$ for any i, k, and $\alpha_i + \alpha_j \neq a_{k_1} + a_{k_2}$ for any i,j, k_1 and k_2, $Y_2(s)$ involves poles which are of first order for the $\lambda_i + \lambda_j \approx \lambda_j$ situation.

Therefore, the approximation to be made is

$$\lambda_1 + \lambda_2 = \lambda_2 \text{ for } \lambda_2 >> \lambda_1 \tag{78}$$

in the expression for $Y_2(s)$. $Y_2(s)$ then consists of simple-order poles. The functional form of $Y_2(s)$ remains the same under the approximation.

The next problem to be addressed is the generation of equations to solve for the unknown $A_{k_1k_2}$ quantities. The general approach is to:

(1) Identify the A_{ii}, $i = 1, \ldots, N$ from residues of poles at $s = -2\lambda_i$

(2) Identify the $A_{\ell m}, \ell \neq m$, $\ell, m = 1, \ldots, N$ from residues of poles at $s = -(\lambda_\ell + \lambda_m)$.

(3) Identify the remaining $A_{k_1k_2}$ from residues of poles $s = -\alpha_i - \lambda_{k_2}$, $i, k_2 = 1, \ldots, N$.

The approximation implies that there are some $A_{\ell m}$, $\ell \neq m$, ℓ, $m = 1, \ldots, N$ that cannot be identified from the poles $\lambda_\ell + \lambda_m$ since $\lambda_\ell + \lambda_m \simeq \lambda_m$ for some ℓ, m combination. This requires the identification technique to be modified to allow generation of the appropriate equations to solve for the unknown $A_{k_1k_2}$. The derivation of a new technique to generate equations is addressed below. In general, there are $(3N^2/2) + (N/2)$ nonzero $A_{k_1k_2}$ to be determined. The poles of $Y_2(s)$ are given by

$$s = a_{k_1} + a_{k_2}, \; k_1 = 1, \ldots, M; \; k_2 = 1, \ldots, N$$

$$s = -\alpha_i + a_{k_2}, \; i = 1, \ldots, L; \; k_2 = 1, \ldots, N$$

$$s = -\alpha_i - \alpha_j, \; i, j = 1, \ldots, L. \tag{79}$$

First, consider poles of the form $s = a_{k_1} + a_{k_2} = 2\lambda_\ell$;

$\ell = 1, \ldots, N$. The terms in $Y_2(s)$ corresponding to the pole at $2\lambda_\ell$ are given by

$$Y_{2\ell\ell}(s) = \sum_{i=1}^{L} \sum_{j=1}^{L} A_{\ell\ell} \frac{1}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)} \frac{1}{s - 2\lambda_\ell} \tag{80}$$

Let the residue of the pole at $2\lambda_\ell$, as obtained by the pencil-of-functions method, be denoted by $\beta_{\ell\ell}$. It follows that

$$A_{\ell\ell} = \beta_{\ell\ell} \left[\sum_{i=1}^{L} \sum_{j=1}^{L} \frac{1}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)} \right]^{-1} \quad \ell = 1, \ldots, N. \tag{81}$$

This procedure results in identification of N of the coefficients, and is unaffected by the approximation.

The general procedure at this point is to consider the poles of the form $s = a_{k_1} + a_{k_2} = \lambda_\ell + \lambda_m$ where $\ell \neq m$ and $\ell, m = 1, \ldots, N$. Since $A_{\ell m} = A_{m\ell}$ for $\ell, m \leq N$, the terms in $Y_2(s)$ corresponding to the pole at $\lambda_\ell + \lambda_m$ are given by

$$Y_{2\ell m}(s) = \sum_{i=1}^{L} \sum_{j=1}^{L} A_{\ell m}$$

$$\cdot \left[\frac{\alpha_i + \alpha_j + 2\lambda_\ell}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)}\right.$$

$$\left. + \frac{\alpha_i + \alpha_j + 2\lambda_m}{(\alpha_j + \lambda_m)(\alpha_i + \lambda_m)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)} \frac{1}{s - \lambda_\ell - \lambda_m}\right] . \quad (82)$$

Let the residue of the pole at $\lambda_\ell + \lambda_m$, as calculated from the pencil-of-functions method, be denoted by $\beta_{\ell m}$. It follows that

$$A_{\ell m} = \beta_{\ell m} \left\{ \sum_{i=1}^{L} \sum_{j=1}^{L} \left[\frac{\alpha_i + \alpha_j + 2\lambda_\ell}{(\alpha_j + \lambda_\ell)(\alpha_i + \lambda_\ell)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)} \right.\right.$$

$$\left.\left. + \frac{\alpha_i + \alpha_j + 2\lambda_m}{(\alpha_j + \lambda_m)(\alpha_i + \lambda_m)(\alpha_i + \alpha_j + \lambda_\ell + \lambda_m)} \right] \right\}^{-1}_{\substack{\ell, m = 1, \ldots, N \\ \ell \neq m, \ell < m}} . \quad (83)$$

This procedure results in identification of $N(N - 1)/2$ of the coefficients provided $\lambda_\ell + \lambda_m \neq \lambda_m$ for any $m, \ell = 1, \ldots, N$.

When $\lambda_\ell + \lambda_m \simeq \lambda_m$ is the situation, specific $A_{\ell m}$, $\ell, m = 1, \ldots, N$, $\ell \neq m$, cannot be identified in this fashion.

Assume that there are K combinations of ℓ, m, $\ell, m = 1, \ldots, N$ such that $\lambda_\ell + \lambda_m \simeq \lambda_m$. There remain $N^2 + K$ unknown $A_{k_1 k_2}$ quantities to be identified. If the input consists of N decaying exponentials, the residues of the poles $s = -\alpha_i - \lambda_j$, $i, j = 1, \ldots, N$ can be used to identify $N^2 - 2K$ unknown $A_{k_1 k_2}$ quantities. Also identified are K quantities which consist of sums of unknown $A_{k_1 k_2}$ quantities. This is illustrated in the following identification technique example.

Consider a nonlinear system whose linear impulse response is given by:

$$h_1(t) = \sum_{i=1}^{3} R_i e^{\lambda_i t} \quad t \geq 0 \tag{84}$$

$$\text{Re}\{\lambda_i\} < 0.$$

The problem is to completely specify the parameters of $h_2(t_1,t_2)$ where

$$h_2(t_1,t_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_1 + a_{k_2}t_2} U(t_2 - t_1)$$

$$+ \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_2 + a_{k_2}t_1} U(t_1 - t_2). \tag{85}$$

The parameters are M, N, a_{k_1}, a_{k_2} and $A_{k_1k_2}$.

The number of residues, $A_{k_1k_2}$, that must be determined is given by

$$W = N M - \frac{N(N-1)}{2} = 27. \tag{86}$$

We know from previous analysis that

$$s = N(N-1)^2 = 3(2)^2 = 12 \tag{87}$$

elements of the set $\{a_{k_1} + a_{k_2}\}$ have zero residues. The elements of $\{a_{k_1} + a_{k_2}\} = \{a'_{k_1k_2}\}$ which have zero residues have the form $\lambda_i + \lambda_j - \lambda_k$, $i \neq k$; $j \neq k$. Identifying those elements of $\{a_{k_1k_2}\}$ that have this form yields the following zero value $A_{k_1k_2}$ quantities

$$
\begin{aligned}
A_{51} &= 0\\
A_{61} &= 0\\
A_{81} &= 0\\
A_{10,1} &= 0\\
A_{62} &= 0\\
A_{72} &= 0\\
A_{82} &= 0\\
A_{92} &= 0\\
A_{53} &= 0\\
A_{73} &= 0\\
A_{93} &= 0\\
A_{10,3} &= 0
\end{aligned}
\tag{88}
$$

We can group all the remaining elements of $\{a'_{k_1k_2}\}$ to specify all the distinct elements of $\{a'_{k_1k_2}\}$. In this group below we include the (k_1,k_2) pairs that produce the given sum $a_{k_1} + a_{k_2}$.

$a_{k_1} + a_{k_2}$	(k_1,k_2)
λ_1	(4,1) (5,2) (6,3)
λ_2	(7,1) (4,2) (8,3) (3,1) (1,3) (2,1) (1,2)
λ_3	(9,1) (10,2) (4,3) (2,3) (3,2)
$2\lambda_1$	(1,1)
$2\lambda_2$	(2,2)
$2\lambda_3$	(3,3)

(89)

where it has been assumed that

$$
\begin{aligned}
\lambda_1 + \lambda_2 &= \lambda_2\\
\lambda_1 + \lambda_3 &= \lambda_3\\
\lambda_2 + \lambda_3 &= \lambda_2
\end{aligned}
$$

We now demonstrate how the equations relating to $A_{k_1k_2}$ are obtained for the set of poles at $(a_{k_1} + a_{k_2})$. Assume that the method of system identification is applied and all appropriate measurements are made. This means that a residue value is available for each distinct element in the set $(a_{k_1} + a_{k_2})$.

We now consider the situation that arises for L = 3. The resultant expression for $Y_2(s)$ is given by

$$Y_2(s) = \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} A_{k_1k_2}$$

$$\cdot \left\{ \left[\frac{\alpha_i + \alpha_j + 2a_{k_1}}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_1})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \right] \right.$$

$$\cdot \left[\frac{1}{s - (a_{k_1} + a_{k_2})} \right] - \left[\frac{1}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_2})} \right] \left[\frac{1}{s + \alpha_j - a_{k_2}} \right]$$

$$- \left[\frac{1}{(\alpha_i + a_{k_1})(\alpha_j + a_{k_2})} \right] \left[\frac{1}{s + \alpha_i - a_{k_2}} \right]$$

$$\left. + \frac{(\alpha_i + \alpha_j + 2a_{k_2})}{(\alpha_j + a_{k_2})(\alpha_i + a_{k_2})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \frac{1}{s + \alpha_i + \alpha_j} \right\} . \tag{90}$$

The quantities $A_{k_1k_2}$, for $k_1 = k_2$, $k_1, k_2 = 1, \ldots, N$ are identified from the poles $s = -2\lambda_{k_1}$, $k_1 = 1, \ldots, N$. This identifies A_{11}, A_{22}, A_{33}.

By considering each set of poles separately, equation (90) can be expressed as three terms

$$Y_{2_1}(s) = \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} A_{k_1k_2}$$

$$\cdot \left[\frac{\alpha_i + \alpha_j + 2a_{k_2}}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_1})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \right] \frac{1}{s - (a_{k_1} + a_{k_2})}$$

$$= \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} A_{k_1k_2} \left\{ \left[\frac{2}{(\alpha_1 + a_{k_1})(2\alpha_1 + a_{k_1} + a_{k_2})} \right] \right.$$

$$+ \left[\frac{2(\alpha_1 + \alpha_2 + 2a_{k_1})}{(\alpha_2 + a_{k_1})(\alpha_1 + a_{k_1})(\alpha_1 + \alpha_2 + a_{k_1} + a_{k_2})} \right]$$

$$+ \left[\frac{2}{(\alpha_2 + a_{k_1})(2\alpha_2 + a_{k_1} + a_{k_2})} \right]$$

$$+ \left[\frac{2(\alpha_1 + \alpha_3 + 2a_{k_1})}{(\alpha_3 + a_{k_1})(\alpha_1 + a_{k_1})(\alpha_1 + \alpha_3 + a_{k_1} + a_{k_2})} \right]$$

$$+ \left[\frac{2}{(\alpha_3 + a_{k_1})(2\alpha_3 + a_{k_1} + a_{k_2})} \right]$$

$$\left. + \left[\frac{2(\alpha_2 + \alpha_3 + 2a_{k_1})}{(\alpha_3 + a_{k_1})(\alpha_2 + a_{k_1})(\alpha_2 + \alpha_3 + a_{k_1} + a_{k_2})} \right] \right\}$$

$$\cdot \left[\frac{1}{s - (a_{k_1} + a_{k_2})} \right] \qquad (91)$$

$$Y_{2_4}(s) = \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} A_{k_1k_2}$$

$$\cdot \left[\frac{\alpha_i + \alpha_j + 2a_{k_2}}{(\alpha_j + a_{k_2})(\alpha_i + a_{k_2})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})} \right]$$

$$\cdot \left(\frac{1}{s + \alpha_i + \alpha_j} \right)$$

$$= \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} A_{k_1 k_2} \left\{ \left[\frac{2}{(\alpha_1 + a_{k_2})(2\alpha_1 + a_{k_1} + a_{k_2})} \right] \right.$$

$$+ \left[\frac{2(\alpha_1 + \alpha_2 + a_{k_2})}{(\alpha_2 + a_{k_2})(\alpha_1 + a_{k_2})(\alpha_1 + \alpha_2 + a_{k_1} + a_{k_2})} \right]$$

$$+ \left[\frac{2(\alpha_2 + \alpha_3 + a_{k_2})}{(\alpha_3 + a_{k_2})(\alpha_2 + a_{k_2})(\alpha_2 + \alpha_3 + a_{k_1} + a_{k_2})} \right]$$

$$\left. + \left[\frac{2}{(\alpha_3 + a_{k_2})(2\alpha_3 + a_{k_1} + a_{k_2})} \right] \right\} \left(\frac{1}{s + \alpha_i + \alpha_j} \right) \qquad (92)$$

And,

$$Y_{2_{2+3}}(s) = - \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} A_{k_1 k_2} \left\{ \left[\frac{1}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_2})} \right] \right.$$

$$\cdot \left(\frac{1}{s + \alpha_j - a_{k_2}} \right) + \left[\frac{1}{(\alpha_j + a_{k_2})(\alpha_i + a_{k_1})} \right]$$

$$\left. \cdot \left(\frac{1}{s + \alpha_i - a_{k_2}} \right) \right\} \qquad (93)$$

Expanding equation (93) yields

$$Y_{2_{2+3}}(s) = \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} A_{k_1 k_2} \left\{ \left[\frac{2}{(\alpha_1 + a_{k_1})(\alpha_1 + a_{k_2})} \right] \frac{1}{s + \alpha_1 - a_{k_2}} \right.$$

$$+ \left[\frac{2}{(\alpha_2 + a_{k_1})(\alpha_1 + a_{k_2})} \right] \frac{1}{s + \alpha_2 - a_{k_2}}$$

$$+ \left[\frac{2}{(\alpha_3 + a_{k_1})(\alpha_1 + a_{k_2})} \right] \frac{1}{s + \alpha_3 - a_{k_2}}$$

$$
\begin{aligned}
&+ \left[\frac{2}{(\alpha_1 + a_{k_1})(\alpha_2 + a_{k_2})}\right] \frac{1}{s + \alpha_1 - a_{k_2}} \\
&+ \left[\frac{2}{(\alpha_2 + a_{k_1})(\alpha_2 + a_{k_2})}\right] \frac{1}{s + \alpha_2 - a_{k_2}} \\
&+ \left[\frac{2}{(\alpha_3 + a_{k_1})(\alpha_2 + a_{k_2})}\right] \frac{1}{s + \alpha_3 - a_{k_2}} \\
&+ \left[\frac{2}{(\alpha_1 + a_{k_1})(\alpha_3 + a_{k_2})}\right] \frac{1}{s + \alpha_1 - a_{k_2}} \\
&+ \left[\frac{2}{(\alpha_2 + a_{k_1})(\alpha_3 + a_{k_2})}\right] \frac{1}{s + \alpha_2 - a_{k_2}} \\
&+ \left.\left[\frac{2}{(\alpha_3 + a_{k_1})(\alpha_3 + a_{k_2})}\right] \frac{1}{s + \alpha_3 - a_{k_2}}\right\} \qquad (94)
\end{aligned}
$$

We now focus attention on equation (94) which can be simplified to obtain

$$
\begin{aligned}
Y_{2_{2+3}}(s) = -2 \sum_{k_1=1}^{10} \sum_{k_2=1}^{3} A_{k_1 k_2} \Bigg\{ & \left[\frac{1}{\alpha_1 + a_{k_1}} \left(\frac{1}{\alpha_1 + a_{k_2}}\right.\right. \\
& \left.\left. + \frac{1}{\alpha_2 + a_{k_2}} + \frac{1}{\alpha_3 + a_{k_2}}\right)\right] \left(\frac{1}{s + \alpha_1 - a_{k_2}}\right) \\
& + \left[\frac{1}{(\alpha_2 + a_{k_1})} \left(\frac{1}{\alpha_1 + a_{k_2}} + \frac{1}{\alpha_2 + a_{k_2}} + \frac{1}{\alpha_3 + a_{k_2}}\right)\right]
\end{aligned}
$$

$$\frac{1}{s+\alpha_2-a_{k_2}} + \left[\frac{1}{(\alpha_3+a_{k_1})}\left(\frac{1}{\alpha_1+a_{k_2}}+\frac{1}{\alpha_2+a_{k_2}}\right.\right.$$

$$\left.\left.+\frac{1}{\alpha_3+a_{k_2}}\right)\right] \quad \left.\frac{1}{s+\alpha_3-a_{k_2}}\right\} \qquad (95)$$

The previous analysis has shown that the unknowns involved in equation (95) are

$$A_{41},\ A_{42},\ A_{43},\ A_{52},\ A_{63},\ A_{71},\ A_{83},\ A_{91},\ A_{10,2},\ A_{21},\ A_{31},\ A_{32}$$

If we consider the pole at $s = -\alpha_1 + a_1$ and follow the procedure detailed previously, we obtain the equation from (95)

$$A_{41}\left[\frac{1}{\alpha_1+a_4}\left(\frac{1}{\alpha_1+a_1}+\frac{1}{\alpha_2+a_1}+\frac{1}{\alpha_3+a_1}\right)\right]$$

$$+ A_{21}\left[\frac{1}{\alpha_1+a_2}\left(\frac{1}{\alpha_1+a_1}+\frac{1}{\alpha_2+a_1}+\frac{1}{\alpha_3+a_1}\right)\right]$$

$$+ A_{71}\left[\frac{1}{\alpha_1+a_7}\left(\frac{1}{\alpha_1+a_1}+\frac{1}{\alpha_2+a_1}+\frac{1}{\alpha_3+a_1}\right)\right]$$

$$+ A_{31}\left[\frac{1}{\alpha_1+a_3}\left(\frac{1}{\alpha_1+a_1}+\frac{1}{\alpha_2+a_1}+\frac{1}{\alpha_3+a_1}\right)\right]$$

$$+ A_{91}\left[\frac{1}{\alpha_1+a_9}\left(\frac{1}{\alpha_1+a_1}+\frac{1}{\alpha_2+a_1}+\frac{1}{\alpha_3+a_1}\right)\right] = \bar{D}_1 \qquad (96)$$

which reduces to

$$\frac{A_{41}}{\alpha_1 + a_4} + \frac{A_{21}}{\alpha_1 + a_2} + \frac{A_{71}}{\alpha_1 + a_7} + \frac{A_{31}}{\alpha_1 + a_3} + \frac{A_{91}}{\alpha_1 + a_9} = \bar{D}_1' \tag{97}$$

Similarly the pole at $s = -\alpha_2 + a_1$ yields the equation

$$A_{41}\left[\frac{1}{(\alpha_2 + a_4)}\right] + A_{21}\left[\frac{1}{\alpha_2 + a_2}\right] + A_{71}\left[\frac{1}{\alpha_2 + a_7}\right] + A_{31}\left[\frac{1}{\alpha_2 + a_3}\right] + A_{91}\left[\frac{1}{(\alpha_2 + a_9)}\right] = \bar{D}_2' \tag{98}$$

Similarly the pole at $s = -\alpha_3 + a_1$ yields the equation

$$A_{41}\left[\frac{1}{(\alpha_3 + a_4)}\right] + A_{21}\left[\frac{1}{\alpha_3 + a_2}\right] + A_{71}\left[\frac{1}{(\alpha_3 + a_7)}\right] + A_{31}\left[\frac{1}{\alpha_3 + a_3}\right] + A_{91}\left[\frac{1}{\alpha_3 + a_9}\right] = \bar{D}_3' \tag{99}$$

From the previous analysis

$$\begin{aligned} a_4 &= 0 & a_2 &= \lambda_2 \\ a_7 &= \lambda_2 - \lambda_1 & a_3 &= \lambda_3 \\ a_9 &= \lambda_3 - \lambda_1 \end{aligned} \tag{100}$$

From the constraints on the analysis, it is known that

$$a_4 \neq a_7 \neq a_9$$

and that our approximation requires

$$a_9 = a_3, \; a_7 = a_2$$

Equation (100) reduces equations (97) and (99) in matrix form, to

$$\begin{bmatrix} \frac{1}{\alpha_1} & \frac{1}{\alpha_1 + a_7} & \frac{1}{\alpha_1 + a_9} \\ \frac{1}{\alpha_2} & \frac{1}{\alpha_2 + a_7} & \frac{1}{\alpha_2 + a_9} \\ \frac{1}{\alpha_3} & \frac{1}{\alpha_3 + a_7} & \frac{1}{\alpha_3 + a_9} \end{bmatrix} \begin{bmatrix} A_{41} \\ A_{71} + A_{21} \\ A_{91} + A_{31} \end{bmatrix} = \begin{bmatrix} \bar{D}_1{}' \\ \bar{D}_2{}' \\ \bar{D}_3{}' \end{bmatrix} \tag{101}$$

Previous analysis (Reference 3) has shown that this set of equations is linearly independent and therefore can be solved for

A_{41}, $(A_{71} + A_{21})$ and $(A_{91} + A_{31})$

If this process is continued by considering the pole at $s = -\alpha_1 + a_2$, an equation similar to equation (97) is obtained, namely with the α_1 quantity replaced by α_2. Similarly, this occurs for the equations (98) and (99). Under these conditions, equation (101) becomes

$$\begin{bmatrix} \frac{1}{\alpha_1} & \frac{1}{\alpha_1 + a_5} & \frac{1}{\alpha_1 + a_1} & \frac{1}{\alpha_1 + a_3} \\ \frac{1}{\alpha_2} & \frac{1}{\alpha_2 + a_5} & \frac{1}{\alpha_2 + a_1} & \frac{1}{\alpha_2 + a_3} \\ \frac{1}{\alpha_3} & \frac{1}{\alpha_3 + a_5} & \frac{1}{\alpha_3 + a_1} & \frac{1}{\alpha_3 + a_1} \end{bmatrix} \begin{bmatrix} A_{42} \\ A_{52} + A_{10,2} \\ A_{12} \\ A_{32} \end{bmatrix} = \begin{bmatrix} \bar{D}_4{}' \\ \bar{D}_5{}' \\ \bar{D}_6{}' \end{bmatrix} \tag{102}$$

If $L = 4$ in this case the above set of equations would be linearly independent and solvable for

A_{42}, A_{12}, A_{32}, and $(A_{52} + A_{10,2})$

Knowing A_{12} (also A_{21}) permits identification of A_{71} from previous analysis.

By considering the pole locations $s = -\alpha_i + a_3$, $i = 1, 2, 3$, the following matrix equation, similar to (101) and (102) is obtained:

$$\begin{bmatrix} \frac{1}{\alpha_1} & \frac{1}{\alpha_1 + a_6} & \frac{1}{\alpha_1 + a_8} & \frac{1}{\alpha_1 + a_3} \\ \frac{1}{\alpha_2} & \frac{1}{\alpha_2 + a_6} & \frac{1}{\alpha_2 + a_8} & \frac{1}{\alpha_2 + a_3} \\ \frac{1}{\alpha_3} & \frac{1}{\alpha_3 + a_6} & \frac{1}{\alpha_3 + a_8} & \frac{1}{\alpha_3 + a_3} \end{bmatrix} \begin{bmatrix} A_{43} \\ A_{63} \\ A_{83} \\ A_{13} \end{bmatrix} = \begin{bmatrix} \bar{D}_7' \\ \bar{D}_8' \\ \bar{D}_9' \end{bmatrix} \tag{103}$$

If L = 4 in this case, these equations are linearly independent and solvable for

$$A_{43},\ A_{63},\ A_{83},\ A_{13}$$

A problem may arise if $\alpha_i + a_3 \cong \alpha_i - a_3$ $(a_6 = -a_3)$ for all i. If this does not occur, then all the $A_{k_1k_2}$ quantities have been identified except for A_{52}, $A_{10,2}$. The sum $A_{52} + A_{10,2}$ has been identified. However, A_{52} can be found from the residue of the pole at $s = -\lambda_1$. This portion of the response is given by equation (91). The only unknown in this equation is A_{52}. Once A_{52} is known, $A_{10,2}$ is also known. Therefore all the unknown $A_{k_1k_2}$ quantities have been identified.

The problem at this point is to demonstrate that this can be achieved for all N.

We now attempt to develop a technique which works in general for all values of N. For convenience we order the λ_i, $i = 1,\ldots,N$, of the linear impulse response such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \cdots < \lambda_N$$

The identification technique will yield a set of equations at pole $s = -\alpha_j + \lambda_1$ given by

$$\sum_{s=1}^{N+K'} \frac{A_{k_{1s}1}}{\alpha_j + a_{k_{1s}}} = D_1 \tag{104}$$

where D_1 is the residue of pole $s = -\alpha_j + \lambda_1$ with all known quantities subtracted out. The index k_{1s} corresponds to those values of k_1 for which $A_{k_1k_2}$ is unknown.

The N + K' $a_{k_{1s}}$ quantities are given by

$$
\begin{aligned}
a_{k_{1s}} &= 0 \\
&= \lambda_2 - \lambda_1 \\
&= \lambda_3 - \lambda_1 \\
&\;\;\vdots \\
&= \lambda_N - \lambda_1 \\
&= \lambda_j \quad \text{K' of these values,} \\
&\qquad j = N, N-1, \ldots, N-K'+1.
\end{aligned}
\tag{105}
$$

The λ_j entries correspond to those poles that arise because of the pole approximation. For example, if $\lambda_2 + \lambda_1 \approx \lambda_1$, then $\lambda_1 = a_{k_{1s}}$ in the above set.

Similarly for α_2, we obtain

$$
\sum_{s=1}^{N+K''} \left(\frac{A_{k_{1s}2}}{\alpha_j + a_{k_{1s}}} \right) = D_2 \tag{106}
$$

where

$$
K'' < K'
$$

and

$$
\begin{aligned}
a_{k_{1s}} &= 0 \\
&= \lambda_1 - \lambda_2 \\
&= \lambda_3 - \lambda_2 \\
&\;\;\vdots \\
&= \lambda_N - \lambda_2 \\
&= \lambda_j \} \text{ K'' of these values, } j = N, N-1, \ldots, N-K''-1.
\end{aligned}
$$

Continuing this process yields

$$\sum_{s=1}^{N+K^N} \left(\frac{A_{k_{1s}N}}{\alpha_j + a_{k_{1s}}}\right) = D_N \tag{107}$$

where

$$K^N \leq K^{N-1} \leq \ldots \leq K'' \leq K'$$

and

$$
\begin{aligned}
a_{k_{1s}} &= 0 \\
&= \lambda_1 - \lambda_N \\
&\quad \lambda_2 - \lambda_N \\
&\quad \vdots \\
&\quad \lambda_{N-1} - \lambda_N \\
&\quad \lambda_j \; \} \; j = N,\ N-1,\ \ldots,\ N - K^N + 1
\end{aligned}
$$

Solution of these equations will result in identification of a number of the $A_{k_1k_2}$ quantities. However several $A_{k_1k_2}$ coefficients will combine together, and only a sum of coefficients is obtained. This is illustrated below.

Assume that $K' = 1$. This implies that

$\lambda_1 + \lambda_N \simeq \lambda_N$ and $\lambda_1 + \lambda_i \not\simeq \lambda_i$ for $i = 1, \ldots, N - 1$.

The equations derived are

$$\sum_{s=1}^{N+1} \frac{A_{k_{1s}1}}{\alpha_j + a_{k_{1s}}} = D_1 \tag{108}$$

where

$$a_{k_{1s}} = 0$$
$$= \lambda_2 - \lambda_1$$
$$\vdots$$
$$= \lambda_{N-1} - \lambda_1$$
$$= \lambda_N - \lambda_1 \simeq \lambda_N$$

If L = N, then a set of linear independent equations is generated that results in solution for all the $A_{k_{1s}1}$ quantities except for a pair given by

$$A_{N^2-N+3,1} + A_{N1}$$

For $k_2 = 2$,

$$\sum_{s=1}^{N} \frac{A_{k_{1s}2}}{\alpha_j + a_{k_{1s}}} = D_2 \tag{109}$$

where

$$a_{k_{1s}} = 0$$
$$= \lambda_1 - \lambda_2$$
$$\vdots$$
$$= \lambda_N - \lambda_2$$

These equations are linearly independent and can be solved directly for $A_{k_{1s}2}$. Similarly a solution is obtained for the quantities $A_{k_{1s}3}, \ldots, A_{k_{1s}N-1}$.

For $k_2 = N$, the equations are

$$\sum_{s=1}^{N+1} \frac{A_{k_{1s}N}}{\alpha_j + a_{k_{1s}}} = D_N \tag{110}$$

where

$$
\begin{aligned}
a_{k_{1s}} &= 0 \\
&= \lambda_1 - \lambda_N \simeq -\lambda_N \\
&\vdots \\
&= \lambda_{N-1} - \lambda_N \\
&\quad \lambda_N
\end{aligned}
$$

If L = N + 1, then a set of N + 1 equations in N + 1 unknown is generated that is linearly independent and solvable for the $A_{k_{1s}N}$.

The quantity A_{1N} is identified from the equation involving $A_{k_{1s}N}$. This results in identification of all the unknown $A_{k_1 k_2}$ in $h_2(t_1, t_2)$.

Suppose next that K' = 2, $(\lambda_N + \lambda_1 \simeq \lambda_N,\ \lambda_{N-1} + \lambda_1 \simeq \lambda_{N-1})$. The resultant equations are

$$\sum_{s=1}^{N+2} \left(\frac{A_{k_{1s}1}}{\alpha_j + a_{k_{1s}}} \right) = D_1 \qquad (111)$$

where

$$
\begin{aligned}
a_{k_{1s}} &= 0 \\
&= \lambda_2 - \lambda_1 \\
&= \lambda_3 - \lambda_1 \\
&\vdots \\
&= \lambda_{N-1} - \lambda_1 \simeq \lambda_{N-1} \\
&= \lambda_N - \lambda_1 \simeq \lambda_N
\end{aligned}
$$

Solution of these equations leads to identification of N - 4 $A_{k_1 k_2}$ quantities and the summations

$$A_{N-1,1} + A_{N^2-N+2,1}$$

$$A_{N,1} + A_{N^2-N+3,1}$$

For $k_2 = 2$, the equations are

$$\sum_{s=1}^{N} \left(\frac{A_{k_{1s}2}}{\alpha_j + a_{k_{1s}}}\right) = D_2 \qquad (112)$$

These are solved directly for the A_{k_12}. Similarly for $k_2 = 3, \ldots, N$. This implies that $A_{1,N}$ and $A_{1,N-1}$ are identified provided $N > 2$. Using the symmetry, $A_{k_1k_2} = A_{k_2k_1}$, $k_1, k_2 = 1, \ldots, N$, yields solution for all the $A_{k_1k_2}$.

Assume that $K' = 2$ but, in this case, the poles are such that $\lambda_N + \lambda_1 \simeq \lambda_N$ and $\lambda_N + \lambda_2 \simeq \lambda_N$. The resultant equations are

$$\sum_{s=1}^{N+2} \frac{A_{k_{1s}1}}{\alpha_j + a_{k_{1s}}} = D_1 \qquad (113)$$

where

$$a_{k_{1s}} = 0$$

$$\lambda_2 - \lambda_1$$

$$\vdots$$

$$\lambda_N - \lambda_1 \simeq \lambda_N$$

This yields solution of $N - 2$ $A_{k_{1s}1}$ quantities and the summation

$$A_{N,1} + A_{N^2-N+3,1}$$

For $k_2 = 2$, the equations become

$$\sum_{s=1}^{N+2} \left(\frac{A_{k_{1s}2}}{\alpha_j + a_{k_{1s}}}\right) = D_2 \qquad (114)$$

where

$$A_{k_{1s}} = 0 \quad \begin{matrix} \lambda_1 - \lambda_2 \\ \lambda_3 - \lambda_2 \\ \vdots \\ \lambda_N - \lambda_2 \approx \lambda_N \end{matrix}$$

Solution of these equations yields all of the unknown $A_{k_1 k_2}$ except for the summation

$$A_{N2} + A_{N^2-N+4,2}$$

For $k_2 = N$, the equation becomes

$$\sum_{s=1}^{N-1} \frac{A_{k_{1sN}}}{\alpha_j + A_{k_{1s}}} = D_N \tag{115}$$

where

$$A_{k_{1s}} = 0 \quad \begin{matrix} \lambda_1 - \lambda_N \approx -\lambda_N \\ \lambda_2 - \lambda_N \approx -\lambda_N \\ \vdots \\ \lambda_{N-1} - \lambda_N \end{matrix}$$

The solution of these equations identifies all $A_{k_{1sN}}$ quantities except for the summation,

$$A_{2N,N} + A_{3N-1,N}$$

This yields solution for A_{N2} which implies that the only unknown $A_{k_1 k_2}$ at this point are

$$A_{2N,N} \text{ and } A_{3N-1,N}$$

An equation involving A_{2N} and $A_{3N-1,N}$ can be obtained from the residue of the pole at $s = -2\alpha_1$. This equation has the form

$$\frac{A_{2N,N}}{(\alpha_1 + \lambda_N)(2\alpha_1 + \lambda_1)} + \frac{A_{3N-1,N}}{(\alpha_1 + \lambda_N)(2\alpha_1 + \lambda_2)} = H_1 \tag{116}$$

where H_1 = residue at pole at $s = -2\alpha_1$. This equation can be rewritten as

$$\frac{A_{2N,N}}{2\alpha_1 + \lambda_1} + \frac{A_{3N-1,N}}{2\alpha_1 + \lambda_2} = H_1' \tag{117}$$

The two equations involving $A_{2N,N}$ and $A_{3N-1,N}$ can be represented in matrix form as

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{2\alpha_1 + \lambda_1} & \frac{1}{2\alpha_1 + \lambda_2} \end{bmatrix} \begin{bmatrix} A_{2N,N} \\ A_{3N-1,N} \end{bmatrix} = \begin{bmatrix} D_N' \\ H_1' \end{bmatrix} \tag{118}$$

$$[B]\ [A] = [Z] \tag{119}$$

For linear independence

$$\det B \neq 0 \tag{120}$$

For det B = 0, it is required that

$$\frac{1}{2\alpha_1 + \lambda_2} = \frac{1}{2\alpha_1 + \lambda_1} \tag{121}$$

or $\lambda_2 = \lambda_1$, which is not permitted by assumption.

Therefore, it is possible to solve for all the $A_{k_1k_2}$ quantities even if K'= Z.

Further analysis has not led to a method of demonstrating that the $A_{k_1k_2}$ coefficients can be identified for an arbitrary number of pole pair approximations ($\lambda_i + \lambda_j \approx \lambda_j$). One of the reasons for this is illustrated in the analysis for two pole pair approximations ($K' = 2$). There are many combinations of potential pole paires to achieve $K' = 2$. As K' increases, these combinations increase significantly in number, and it is necessary to show that a set of linearly independent equations can be found. This process rapidly becomes complex to show, <u>in general</u>, that these equations can be generated.

At this point, it should be noted that significantly more equations are generated than are used in the identification process. It is possible that some of these equations can be used to form a linearly independent set of equations to solve for the unknown $A_{k_1k_2}$. The problem is that a method has not been found for demonstrating that a linear independent set of equations involving the unknown $A_{k_1k_2}$ quantities can <u>always</u> be obtained using the identification technique. The algebraic nature of the equations has, to date, prevented linear independence of any set of equations other than those generated from poles at $s = -\alpha_i + \lambda_{k_2}$, $i = 1,\ldots,L$; $k_2 = 1,\ldots,N$ from being proved in general.

The above analysis implies that the pole approximation ($\lambda_i + \lambda_j \approx \lambda_j$) is a reasonable approach to alleviating the numerical problem associated with poles of this type. For one- or two-pole pairs, it has been shown that the $A_{k_1k_2}$ quantities can be identified. For more than two poles, each system identification problem must be considered individually. Although a general identification technique is not defined here, it is probable that there will be a sufficient number of linear independent equations to solve for all the $A_{k_1k_2}$ quantities. Each system identification problem must be considered individually to find a sufficient number of linearly independent equations.

D. DOMINANT POLE CONCEPT

As noted previously, the identification technique becomes significantly computationally complex as the number of poles of the linear system increases. The second-order impulse response of a weakly nonlinear system is given by

$$h_2(t_1,t_2) = \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_1}t_1 + a_{k_2}t_2} U(t_2 - t_1)$$

$$+ \sum_{k_1=1}^{M} \sum_{k_2=1}^{N} A_{k_1k_2} e^{a_{k_2}t_1 + a_{k_1}t_2} U(t_1 - t_2) \quad (122)$$

For a given system, it is possible that this impulse response is dominated by a limited number of $A_{k_1k_2}$ terms. This implies that $h_2(t_1,t_2)$ can be represented by fewer terms than are given in equation (122). This reduces the number of coefficients that must be identified and will ease the computational problem associated with the identification technique. This section investigates how this approach may be implemented to reduce the order of the second-order impulse response.

The second-order system response, $Y_2(s)$, has been shown to be given by equation (19). The primary focus on reducing the order of the identification problem must be on reducing the order of $Y_2(s)$. This is because the order of the residue equation $R = C^{-1} Y$ is directly determined by the order of $Y_2(s)$.

All of the poles of $Y_2(s)$ do not contribute equally to the system output, $y_2(t)$. It is possible that some poles of $Y_2(s)$ contribute negligibly to the output. If these poles can be identified, then the second-order response, $Y_2(s)$, can be approximated by a response $Y_2'(s)$, where $Y_2'(s)$ does not contain the poles of $Y_2(s)$ that negligibly impact the output.

The basic problem with identifying which poles of $Y_2(s)$ have a negligible impact on the output is that this requires knowledge of the $A_{k_1k_2}$ quantities that we are trying to identify. This implies that the poles of $Y_2(s)$ that have a minor impact on the system output must be identified by other than analytical means. However, it is unlikely that this can be accomplished.

A potential method of reducing the order of $Y_2(s)$ by identifying the negligible poles is presented here. The number of poles and the location of each pole of $Y_2(s)$ are known from identification of the linear transfer function, $H_1(s)$. The proposed technique is as follows. First, the nonlinear system is excited by the input $x(t) = e^{-\alpha_1 t}$. The resultant output, $Y_2(s)$, contains $N[(N + 5)/2] + 1$ poles. The normal procedure is to integrate the input and output N' times and solve the resultant equations for the system residues. The suggested procedure is to assume that the number of significant poles of $Y_2(s)$ is $N'' < N'$. The input and output responses are integrated N'' times, the appropriate inner products are formed and the resultant equation for the pole locations is solved. If these pole locations agree with N'' of the predicted pole locations, it can be assumed that an N'' pole approximation of $Y_2(s)$ is a valid representation. If the pole locations do not agree, then N'' is increased by 1 and the above process is repeated. This is continued until there is good correlation between the identified poles and those predicted from $H_1(s)$.

In order to demonstrate that such a technique is feasible, consider the following example. Suppose we have a system with three poles, where

$$H(s) = \frac{2.8069192 \times 10^5}{s + 0.011550998(2\pi)(10^6)} - \frac{2.7368441 \times 10^8}{s - 10.6161986(2\pi)(10^6)} + \frac{R_3}{s - 1.25(2\pi)(10^6)} \qquad (123)$$

The residue, R_3, will be left undefined for the moment. This transfer function was inserted into the computer simulation of the identification technique. (This simulation was described in detail in Reference 1.)

The simulation assumed that the system of interest was a two-pole system. The residue value, R_3, was varied in amplitude to determine under what conditions H(s) could be accurately represented by a two-pole system.

The results of the simulation are given in Table 5. These results indicate that for $R_3 \leq 10^4$, identification technique identifies the other poles and residues of H(s) with less than 0.25 percent error. For these cases, $R_3/2.8069192 \times 10^5 < 0.035$ and $R_3/2.7368441 \times 10^8 < 3.6 \times 10^{-5}$, which indicates that the poles of H(s) at $s = -0.011550998(2\pi) \times 10^6$ and $s = -10.616986(2\pi) \times 10^6$ dominate the transfer function. As the residue R_3 approaches and exceeds the magnitude of the residue of the pole at $3 = -0.01150998(2\pi)(10^6)$, the identification technique degrades in performance as it attempts to identify the two-pole model of H(s). The reason for this is that, as R_3 increases, H(s) is not dominated by the two poles; therefore, the proposed technique is no longer a valid approach to identifying H(s).

Another approach to reducing the order of $Y_2(s)$ and subsequently easing the computational problem is to apply a dominant pole concept to the linear portion of the system. The linear transfer function of the system of interest is assumed to be of the form

$$H_1(s) = \sum_{i=1}^{N} \frac{R_i}{s - \lambda_i} .$$

It is possible that $H_1(s)$ is dominated by several of the N poles. For example, suppose

TABLE 5. IDENTIFICATION TECHNIQUE PERFORMANCE - TWO-POLE APPROXIMATION OF A THREE POLE SYSTEM

Magnitude of Residue, R_3	Predicted System Poles (MHz)	Percentage Error	Predicted System Residues	Percentage Error
0	0.011545154	-0.02	2.8063116×10^5	-0.021
	10.615088	-0.0178	-2.7362615×10^8	-0.05
10^2	0.011553882	0.11	2.80725×10^5	-0.011
	10.604596	-0.11	-2.7336985×10^8	0.025
10^3	0.011550467	-0.094	2.8068819×10^5	-0.00133
	10.606755	-0.096	-2.7342789×10^8	-0.0046
10^4	0.011551757	-0.21	2.807005×10^5	0.003
	10.590731	-0.25	-2.7309232×10^8	0.006
10^5	0.011517682	-0.972	2.8031985×10^5	-0.132
	10.483458	-1.25	-2.7102316×10^8	-0.288
10^6	0.011223832	-8.04	2.7703829×10^5	-1.3
	9.4884792	-10.63	-2.5167507×10^8	-8.1

$$H_1(s) = \frac{R_1}{s - \lambda_1} + \frac{R_2}{s - \lambda_2}$$

$$= (R_1 + R_2)\left[\frac{s - \left(\frac{R_1 \lambda_2 + R_2 \lambda_1}{R_1 + R_2}\right)}{(s - \lambda_1)(s - \lambda_2)}\right] \tag{124}$$

If

$$\frac{R_1 \lambda_2 + R_2 \lambda_1}{R_1 + R_2} \simeq \lambda_1$$

or if

$$\frac{R_1 \lambda_2 + R_2 \lambda_1}{R_1 + R_2} \simeq \lambda_2 \tag{125}$$

then $H_1(s)$ could be represented by

$$H_1(s) = \frac{(R_1 + R_2)}{s - \lambda_2}$$

or

$$H_1(s) = \frac{(R_1 + R_2)}{s - \lambda_1} \tag{126}$$

respectively. In these cases, the two-pole linear transfer function $H_1(s)$ can be represented by a single-pole transfer function.

This is significant because the order of $Y_2(s)$ varies approximately with N^2, so that any reduction in N achieves an even greater reduction in the number of poles of $Y_2(s)$.

The impact of this approximation on $H_2(s_1,s_2)$ or $h_2(t_1,t_2)$ is presented below. Consider an exact system representation given by

$$H_1(s) = \sum_{i=1}^{2} \frac{R_i}{s - \lambda_i} \tag{127}$$

The resultant $h_2(t_1,t_2)$ is given by

$$
\begin{aligned}
h_2(t_1,t_2) = \Big[& A_{11}\, e^{\lambda_1(t_1 + t_2)} + A_{12}\, e^{\lambda_1 t_1 + \lambda_2 t_2} \\
& + A_{21}\, e^{\lambda_2 t_1 + \lambda_1 t_2} + A_{22}\, e^{\lambda_2(t_1 + t_2)} \\
& + A_{31}\, e^{\lambda_1 t_2} + A_{32}\, e^{\lambda_2 t_2} \\
& + A_{42}\, e^{(\lambda_1 - \lambda_2)t_1 + \lambda_2 t_2} \\
& + A_{51}\, e^{(\lambda_2 - \lambda_1)t_1 + \lambda_1 t_2} \Big] U(t_2 - t_1) \\
& + \Big[A_{11}\, e^{\lambda_1(t_1 + t_2)} + A_{12}\, e^{\lambda_1 t_2 + \lambda_2 t_1} \\
& + A_{21}\, e^{\lambda_2 t_2 + \lambda_1 t_1} + A_{22}\, e^{\lambda_2(t_1 + t_2)} \\
& + A_{31}\, e^{\lambda_1 t_1} + A_{32}\, e^{\lambda_2 t_1} \\
& + A_{42}\, e^{(\lambda_1 - \lambda_2)t_2 + \lambda_2 t_1} \\
& + A_{51}\, e^{(\lambda_2 - \lambda_1)t_2 + \lambda_1 t_1} \Big] U(t_1 - t_2) \qquad (128)
\end{aligned}
$$

If

$$\frac{R_1 \lambda_2 + R_2 \lambda_1}{R_1 + R_2} \simeq \lambda_2$$

then

$$H_1(s) = \frac{R_1}{s - \lambda_1}$$

The resultant $h_2(t_1,t_2)$ is given by

$$h_2'(t_1,t_2) = \left[A_{11}' \, e^{\lambda_1(t_1 + t_2)} + A_{21}' \, e^{\lambda_1 t_2}\right] U(t_2 - t_1)$$
$$+ \left[A_{11}' \, e^{\lambda_1(t_1 + t_2)} + A_{21}' \, e^{\lambda_1 t_1}\right] U(t_1 - t_2) \qquad (129)$$

The approximate expression for $h_2(t_1,t_2)$ implies that

$$A_{11}' = A_{11}$$
$$A_{21}' = A_{31}$$
$$A_{12} = A_{21} = A_{22} = A_{32} = A_{42} = A_{51} = 0 \qquad (130)$$

for $h_2'(t_1,t_2) = h_2(t_1,t_2)$.

The approximate expression for $h_2(t_1,t_2)$ retains only two of the original $A_{k_1k_2}$ coefficients, meaning that the approximation $(R_1 \lambda_2 + R_2 \lambda_1)/(R_1 + R_2) \doteq \lambda_2$ implies that six of the $A_{k_1k_2}$ coefficients are zero. In order to see how this approximation comes about, we consider the following example, which is a simple single nonlinearity (no-memory) nonlinear system, as shown in Figure 4.

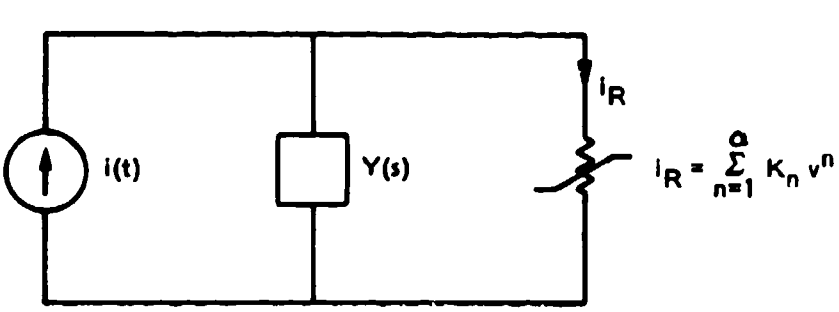


Figure 4. Simple Single Nonlinearity (no-memory) Nonlinear System

It has been shown (Reference 3) that the second-order impulse response of this system has the form

$$
h_2(t_1,t_2) = \begin{cases} -K_2 \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\dfrac{R_l R_i R_j}{\lambda_l - \lambda_i - \lambda_j} \right) \\ \quad \cdot \left[e^{(\lambda_l - \lambda_i)t_1 + \lambda_i t_2} - e^{\lambda_j t_1 + \lambda_i t_2} \right] & t_2 > t_1 \\ -K_2 \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\dfrac{R_l R_i R_j}{\lambda_l - \lambda_i - \lambda_j} \right) \\ \quad \cdot \left[e^{\lambda_i t_1 + (\lambda_l - \lambda_i)t_2} - e^{\lambda_i t_1 + \lambda_j t_2} \right] & t_2 < t_1 \; . \end{cases} \tag{131}
$$

where it has been assumed that the linear impulse response of this system is

$$
h_1(t) = \sum_{i=1}^{N} R_i \, e^{\lambda_i t} \qquad t \geq 0 \; . \tag{132}
$$

For purposes of example, it is assumed that N = 2 and that

$$
h_1(t) = R_1 \, e^{\lambda_1 t} + R_2 \, e^{\lambda_2 t} \; . \tag{133}
$$

If $h_2(t_1,t_2)$ is expanded for N = 2 and put in the standard functional form given by

$$
\begin{aligned} h_2(t_1,t_2) = & \sum_{k_1=1}^{5} \sum_{k_2=1}^{2} A_{k_1 k_2} \, e^{a_{k_1} t_1 + a_{k_2} t_2} \, U(t_2 - t_1) \\ & + \sum_{k_1=1}^{5} \sum_{k_2=1}^{2} A_{k_1 k_2} \, e^{a_{k_2} t_1 + a_{k_1} t_2} \, U(t_1 - t_2) \end{aligned} \tag{134}
$$

where the $A_{k_1 k_2}$ quantities are defined as follows:

$$A_{11} = K_2R_1^{\,2}\left[-\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2 - 2\lambda_1}\right]$$

$$A_{12} = A_{21} = -\,K_2R_1R_2\left[\frac{R_1}{\lambda_2} + \frac{R_2}{\lambda_1}\right]$$

$$A_{22} = K_2R_2^{\,2}\left[-\frac{R_2}{\lambda_2} + \frac{R_1}{\lambda_1 - 2\lambda_2}\right]$$

$$A_{31} = K_2R_1^{\,2}\left[\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2}\right]$$

$$A_{32} = K_2R_2^{\,2}\left[\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2}\right]$$

$$A_{41} = 0$$

$$A_{42} = K_2R_1R_2\left[\frac{R_2}{\lambda_1 - 2\lambda_2} + \frac{R_1}{\lambda_2}\right]$$

$$A_{51} = K_2R_1R_2\left[\frac{R_2}{\lambda_2 - 2\lambda_1} + \frac{R_1}{\lambda_1}\right]$$

$$A_{52} = 0 \qquad (135)$$

The dominant pole assumption was that

$$\frac{R_1\,\lambda_2 + R_2\,\lambda_1}{R_1 + R_2} \simeq \lambda_2\ . \qquad (136)$$

This requires that

$$R_1 >> R_2 \text{ and } R_1\,\lambda_2 >> R_2\,\lambda_1$$

If this assumption is applied to the resultant $A_{k_1k_2}$ quantities, we obtain

$$A_{11} \simeq -\frac{K_2 R_1^{\ 3}}{\lambda_1}$$

$$A_{12} = A_{21} = -K_2 R_1 R_2 \left[\frac{R_1}{\lambda_2} + \frac{R_2}{\lambda_1} \right]$$

$$A_{22} = \frac{K_2 R_2^{\ 3}}{\lambda_1}$$

$$A_{31} = \frac{K_2 R_1^{\ 3}}{\lambda_1}$$

$$A_{32} = \frac{K_2 R_2^{\ 2} R_1}{\lambda_1}$$

$$A_{41} = 0$$

$$A_{42} = K_2 R_1 R_2 \left[\frac{R_2}{\lambda_1 - 2\lambda_2} + \frac{R_1}{\lambda_2} \right]$$

$$A_{51} = \frac{K_2 R_1^{\ 2} R_2}{\lambda_1}$$

$$A_{52} = 0 \qquad (137)$$

For $R_1 >> R_2$, we have

$$A_{11} >> A_{22}$$

$$A_{11} >> A_{12}$$

$$A_{11} >> A_{32}$$

$$A_{11} >> A_{42}$$

$$A_{11} >> A_{51}$$

$$A_{31} >> A_{22}$$

$$A_{31} >> A_{12}$$

$$A_{31} >> A_{32}$$

$$A_{31} >> A_{42}$$

$$A_{31} >> A_{51} \qquad (138)$$

which reduces the second-order impulse response to:

$$h_2(t_1,t_2) = K_2 \left[- \frac{R_1{}^3}{\lambda_1} e^{\lambda_1 t_1 + \lambda_1 t_2} + \frac{R_1{}^3}{\lambda_1} e^{\lambda_1 t_2} \right] U(t_2 - t_1) + K_2 \left[- \frac{R_1{}^3}{\lambda_1} e^{\lambda_1 t_2 + \lambda_1 t_1} + \frac{R_1{}^3}{\lambda_1} e^{\lambda_1 t_1} \right] U(t_2 - t_1) \tag{139}$$

If $h_1(t)$ is approximated by

$$h_1(t) = R_1 e^{\lambda_1 t} \qquad t \geq 0 \tag{140}$$

using the dominant pole concept, the resultant $h_2(t_1,t_2)$ is given by

$$h_2'(t_1,t_2) = \frac{K_2 R_1{}^3}{\lambda_1} \left[e^{\lambda_1 t_2} - e^{\lambda_1 t_1 + \lambda_2 t_2} \right] U(t_2 - t_1) + \frac{K_2 R_1{}^3}{\lambda_1} \left[e^{\lambda_1 t_1} - e^{\lambda_1 t_2 + \lambda_2 t_1} \right] U(t_1 - t_2) \tag{141}$$

or

$$h_2'(t_1,t_2) = \left[A_{11} e^{\lambda_1 t_1 + \lambda_1 t_2} + A_{21} e^{\lambda_1 t_2} \right] U(t_2 - t_1) + \left[A_{11} e^{\lambda_1 t_2 + \lambda_1 t_1} + A_{21} e^{\lambda_1 t_1} \right] U(t_1 - t_2) \tag{142}$$

where

$$A_{11} = - \frac{K_2 R_1{}^3}{\lambda_1}$$

$$A_{21} = \frac{K_2 R_1{}^3}{\lambda_1} \tag{143}$$

It is noted that, given the dominant pole assumption,

$$h_2(t_1,t_2) \approx h_2'(t_1,t_2) \tag{144}$$

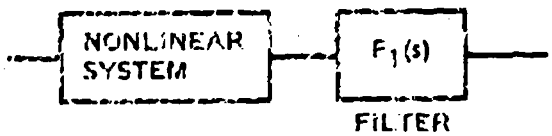
This analysis demonstrates how the approximation in $h_1(t)$ propagates to $h_2(t_1,t_2)$. It supports the approach using the dominant pole approximation on the linear transfer function.

It is noted from the analysis that the dominant pole is present because $H_1(s)$ has a pole and zero which tend to cancel each other in the $s = \sigma + j\omega$ domain. This illustrates why this approach cannot be used on $Y_2(s)$ since the unknown A_{k1k2} quantities prevent the zeros of $Y_2(s)$ from being known. This prohibits association of pole-zero pairs for possible cancellation and subsequent reduction in the order of $Y_2(s)$.

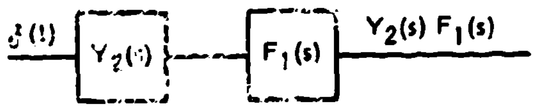
E. RESTRICTED FREQUENCY RANGE CONCEPTS

There are several methods of reducing the order of the second-order response which we classified as restricted frequency range approaches. These approaches basically modify the input or the system output to ease the identification problem.

The primary restricted frequency approach is to use a filter on the output of the system under test. The purpose of the filter is to selectively restrict the system output to a particular frequency range. Consider the example shown below:



For second-order impulse response considerations, the equivalent system is as shown below:



The system output becomes $Y_2(s)\ F_1(s)$. $F_1(s)$ must be band-limited with respect to $Y_2(s)$. The number of poles of $Y_2(s)\,F1(s)$ depends on the number of poles of $Y_2(s)$ within the bandwidth of $F_1(s)$ and the number of poles of $F_1(s)$. For this technique to offer any advantage, the number of poles of $Y_2(s)\ F_1(s)$ must be less than were present before filtering.

The poles of $Y_2(s)$ are of the general form

$$
\begin{array}{ll}
\lambda_i & i = 1,\ldots,N \\
\lambda_\ell + \lambda_m & \ell,m = 1,\ldots,N \\
-\alpha_i + \lambda_j & i,j = 1,\ldots,N \\
-\alpha_i + \alpha_j & i,j = 1,\ldots,N
\end{array}
$$

The α_i are selected by the identification technique user. Proper selection of the α_i can cause the poles of $Y_2(s)$ to bunch up in certain frequency ranges.

Consider, for example, a two-pole system with poles λ_1 and λ_2 ($\lambda_2 > \lambda_1$). The poles of $Y_2(s)$ are

$$
\begin{array}{ll}
\lambda_1 & -2\alpha_1 \\
\lambda_2 & -(\alpha_1 + \alpha_2) \\
\lambda_1 + \lambda_2 & -2\alpha_2 \\
-\alpha_1 + \lambda_1 & 2\lambda_1 \\
-\alpha_1 + \lambda_2 & 2\lambda_2 \\
-\alpha_2 + \lambda_1 & \\
-\alpha_2 + \lambda_2 &
\end{array}
$$

Suppose that

$$
\begin{aligned}
\lambda_2 &= \rho_1\,\lambda_1 \\
\alpha_1 &= \rho_2\,\lambda_1 \\
\alpha_2 &= \rho_3\,\lambda_1
\end{aligned}
\qquad (145)
$$

Then the poles of $Y_2(s)$ are

$$\lambda_1$$

$$\rho_1\lambda_1$$

$$2\lambda_1$$

$$2\rho_1\lambda_1$$

$$(1 + \rho_1)\lambda_1$$

$$(\rho_2 + 1)\lambda_1$$

$$(\rho_2 + \rho_1)\lambda_1$$

$$(\rho_3 + 1)\lambda_1$$

$$(\rho_3 + \rho_1)\lambda_1$$

$$2\rho_2\lambda_1$$

$$(\rho_2 + \rho_3)\lambda_1$$

$$2\rho_3\lambda_1$$

Assume that we select $\rho_1 = 4.5$, $\rho_2 = 1.2$, $\rho_3 = 3.9$, then the poles are

λ_1, $2\lambda_1$, $2.2\lambda_1$, $2.4\lambda_1$, $4.5\lambda_1$, $4.9\lambda_1$, $5.1\lambda_1$, $5.5\lambda_1$, $5.7\lambda_1$,

$7.8\lambda_1$, $8.4\lambda_1$, $9\lambda_1$.

These poles are bunched in essentially two groups:

$(\lambda_1 \rightarrow 2.4\lambda_1)$ and $(4.5\lambda_1 \rightarrow 9\lambda_1)$.

If a filter with a 3-dB bandwidth of approximately $2.4\lambda_1$ is used on the output of the system under test, the resultant output contains the contributions of only a limited number (in this case, four) of the poles of $Y_2(s)$.

The performance of this approach was evaluated using the computer simulation of the identification technique. The system considered was, once again, that given by

$$H(s) = \frac{2.8069192 \times 10^5}{s + 0.011550998(2\pi)(10^6)} - \frac{2.7368441 \times 10^8}{s + 10.616986(2\pi) \times 10^6} \tag{146}$$

The filter transfer function was assumed to be

$$H_A(s) = \frac{1.2 \times 10^7}{s + \gamma_1} . \tag{147}$$

where the pole location γ_1 was left variable. The simulation was to evaluate the poles of a two-pole system representation of $H(s)H_A(s)$, where the two poles to be identified are the two low frequency poles, $-0.011550998(2\pi) \times 10^6$ and γ_1. The pole of interest is that at $s = -0.011550998(2\pi) \times 10^6$ since γ_1 will be selected by the user and will be known.

This procedure was simulated for $\gamma_1 = 0.51\,(2\pi) \times 10^6$ and an input

$$x(t) = e^{-0.2(2\pi) \times 10^6 t}$$

The results are tabulated in Table 6. These results indicate that this approach produces acceptable performance if the integration time is increased above that used for the original system identification. This is to be expected because, in this case, the identification technique is attempting to identify two low frequency poles instead of one low and one high frequency pole; this, in general, will require longer integration times.

This procedure was repeated for an input given by

$$x(t) = e^{-2\pi \times 10^6 t}$$

The results are presented in Table 7. These results basically agree with those of Table 6 and support the need for increased integration time.

TABLE 6. IDENTIFICATION TECHNIQUE PERFORMANCE - RESTRICTED FREQUENCY RANGE APPROACH; FILTER POLE AT s = -0.51 MHz

Integration Time (μs)	Predicted System Pole (MHz)	Percentage Error	Predicted System Residue	Percentage Error
9.6	0.0137455	15.96	2.90046467×10^5	+3.33
28.8	0.0116741579	0.01066	2.7341247×10^5	-2.59

TABLE 7. IDENTIFICATION TECHNIQUE PERFORMANCE - RESTRICTED FREQUENCY RANGE APPROACH; FILTER POLE = 1 MHz

Integration Time (μs)	Predicted System Pole (MHz)	Percentage Error	Predicted System Residue	Percentage Error
9.6	0.0127462611	10.3	2.7850044×10^5	-0.78
28.8	0.011622197	0.616	2.720117229×10^5	-3.09

The identification technique can then be used to evaluate the residues of these poles. Once these residues are known, the technique can be repeated without using the filter. The known portion of $Y_2(s)$ can be subtracted out before processing and the resultant identification problem is reduced to one of lesser order (in this case, 8 instead of 12).

An alternative approach at this point, once 4 of the 12 residues have been identified, is to attempt to select the α_i to separate the remaining poles into distinct groups and use a different filter to limit the number of poles of $Y_2(s)$. This essentially repeats the original identification technique approach but on the reduced order system.

The key to the technique is to use a filter which effectively attenuates the contributions of the poles outside the frequency band of interest. Consider the example shown below. The system response is as shown in Figure 5.

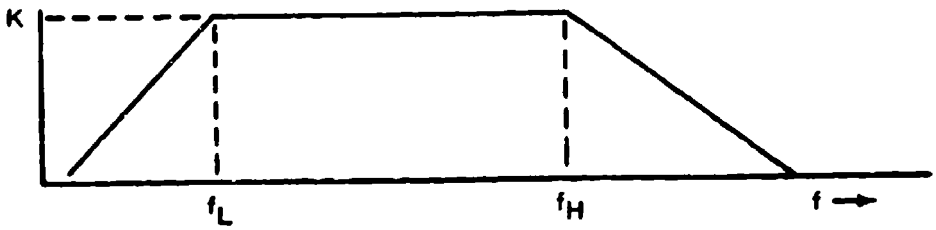


Figure 5. Example System Frequency Response

The filter should have a break frequency equal to or slightly greater than f_L. The amplifier response should be down considerably at frequency f_H to effectively attenuate the frequency response of $Y_2(s)$ above f_H. An attenuation of at least 20 dB seems reasonable for adequate performance of the technique.

Another approach to reducing the order of $Y_2(s)$ for identification purposes is presented here. It has been shown (Reference 2) that the pencil-of-functions identification technique can be modified to divide the frequency spread of the system output into three bands: low, midrange and high. The identification technique is applied by selecting an appropriate input frequency fairly well matched to one of the frequency ranges given above. This procedure is repeated for each frequency range. The total transfer function is obtained by matching the functions at the transition points between frequency ranges and slightly modifying pole locations and gain constants. This technique has been shown (Reference 2) to produce accurate results while reducing the order of the identification problem.

Appropriate selection of integration time for the identification technique can be used in special cases to reduce the order of the identification technique. Consider a two-pole system with poles λ_1 and λ_2, where $\lambda_2 >> \lambda_1$. If a short integration time is used, e.g., $T \approx 1/\lambda_1$, the system output will be impacted very little by the low frequency pole. This is because the contribution from the pole $s = \lambda_1$ is essentially constant over the integration period. This implies that the variation of the system response over the integration period is due to the high frequency pole only.

In order to demonstrate how such a technique would perform, we once again consider the system

$$H(s) = \frac{2.8069192 \times 10^5}{s + 0.011550998(2\pi) \times 10^6} - \frac{2.7368441 \times 10^8}{s + 10.616986(2\pi) \times 10^6} \tag{148}$$

The analysis of Reference 1 (Part I of this study) demonstrated that an integration time of 9.6 μs resulted in generally favorable performance of the identification technique. For this analysis, this integration is varied from 0.0024 μs to 2.4 μs, and the performance of the identification technique is investigated. The simulation is set up to identify a single pole system, in this case, the high frequency pole at $s = -10.616986(2\pi) \times 10^6$.

The results of this simulation are shown in Table 8. The results indicate that, if a short integration time (compared to the reciprocal of the low frequency pole) is used in the identification processing, then the high frequency pole and corresponding residue of H(s) are accurately predicted. The results indicate that at least a 100:1 reduction in integration time from the original 9.6 μs is required to effectively isolate the high frequency pole response. These results suggest that the integration time be less than 1/(high frequency pole) for accurate identification performance.

This integration time approach is related somewhat to the wide-band processing approach of Reference 2. This is a good technique to use on wide-band systems where there are a set of low frequency poles and a set of high frequency poles. Once the high frequency poles and residues are identified, the normal identification procedure is followed and the contributions of the high frequency poles are subtracted out from the system output before pencil-of-functions processing.

TABLE 8. IDENTIFICATION TECHNIQUE PERFORMANCE - RESTRICTED FREQUENCY RANGE VIA CONTROL OF INTEGRATION TIME

Integration Period (μs)	Predicted System Pole (MHz)	Percentage Error	Predicted System Residue	Percentage Error
0.0024	-10.631012	0.132	-2.7340719×10^8	-0.1
0.024	-10.637851	0.196	-2.7345562×10^8	-0.08
0.24	-10.946062	3.1	-2.7879182×10^8	1.86
2.4	-5.498899	48	-1.2121018×10^8	55.7

F. SEPARATION OF RESPONSES

An important requirement for the identification technique is the ability to excite the nonlinear system such that the linear response is isolated from second- and higher-order responses and that the linear plus second-order response is isolated from third- and higher-order responses. This requirement impacts both the feasibility of implementing the identification technique and the computational complexity involved in identifying the nonlinear impulse responses. This critical issue is addressed in detail in this section.

The basic assumption on which the identification technique is founded is that the nonlinear system can be excited in such a manner that the system response is linear. Techniques of validating linear operation of a nonlinear system are addressed in detail in Reference 3. Basically, the nonlinear system is excited by a sinusoidal signal of amplitude A and a spectral analysis of the system response is obtained. Amplitude A is adjusted until the spectral content of the system output shows that the magnitude of second and higher order harmonic frequencies is significantly below that of the fundamental component. This procedure permits determination of the linear impulse response of the nonlinear system, $h_1(t)$. Identification of $h_1(t)$ leads, as has been shown previously, to identification of the natural frequencies of the second and third-order impulse responses. Therefore, it is a key element in the identification process.

If the nonlinear system cannot be excited such that its output response is linear, the identification procedure increases in complexity but the second-order impulse response, $h_2(t_1, t_2)$ can still be identified. This fact is demonstrated below.

Assume that the nonlinear system can be excited such that only $y_1(t) + y_2(t)$ can be isolated from third- and higher-order system responses. Define $y_a(t)$ as

$$y_a(t) = y_1(t) + y_2(t) \tag{149}$$

and

$$Y_a(s) = Y_1(s) + Y_2(s). \tag{150}$$

The poles of $Y_a(s)$ are given by

$$s = a_{k_1} + a_{k_2};\ k_1 = 1, \ldots, M;\ k_2 = 1, \ldots, N$$

$$s = -\alpha_a + a_{k_2};\ i = 1, \ldots, L;\ k_2 = 1, \ldots, N$$

$$s = -\alpha_i - \alpha_j;\ i,j = 1,\ldots,L.$$

$$s = -\alpha_i;\ i = 1,\ldots,L$$

$$s = a_{k_2};\ k_2 = 1,\ldots,N \tag{151}$$

There are $2N(N + 1) + N$ poles if $L = N$, as is generally the case. Since N may not be known, assume that $L = 1$. Then, the number of poles in $Y_a(s)$ is

$$\beta = 2N + 2 + \frac{N(N + 1)}{2} = \frac{N^2}{2} + \frac{5N}{2} + 2 \tag{152}$$

These poles are of the form

$$\lambda_i;\ i = 1,\ldots N$$

$$\lambda_i + \lambda_j;\ i,j = 1,\ldots N$$

$$-\alpha_1 + \lambda_i;\ i = 1,\ldots,N$$

$$-\alpha_1$$

$$-2\alpha_1$$

If the identification process is applied to the response, $y_1(t) + y_2(t)$, then β and the poles of $Y_a(s)$ will be identified. By associating the identified poles with the above list, it will be possible to identify the λ_i, $i = 1,\ldots,N$. The number of poles in the linear system, N, can be found from β.

This association will be done as follows. In the list of identified poles, there will be N pairs of poles having the relationship

$$a_j = 2a_\ell$$
$$a_k = 2a_m \tag{153}$$

where a_j, a_k, a_ℓ, a_m are identified poles. Furthermore there will be identified pole pairs of the form

$$a_n = -\alpha_1 + a_p \tag{154}$$

where α_1 is known from the input. It is noted that sufficient data is available to identify the poles λ_i, $i = 1,\ldots,N$ of the linear system. The residues of the poles of the linear transfer function can be found from the residues of the poles of $Y_a(s)$

corresponding to $s = -\alpha_1$. Then the $A_{k_1k_2}$ quantities of $h_2(t_1,t_2)$ can be found in the normal way.

This is shown here.

Given a nonlinear system with a linear transfer function given by

$$H_1(s) = \sum_{i=1}^{N} \frac{R_i}{s + \alpha_i} \tag{155}$$

and an input X(s) given by

$$X(s) = \sum_{i=1}^{L} \frac{C_i}{s + \alpha_i} \tag{156}$$

the linear response is described by

$$Y_1(s) = \sum_{j=1}^{N} \sum_{i=1}^{L} \left(\frac{R_j C_i}{\alpha_i + a_j}\right) \left(\frac{1}{S - a_j} - \frac{1}{S + \alpha_i}\right) \tag{157}$$

The second-order response is given by

$$\begin{aligned} Y_2(s) = \sum_{k_2=1}^{M} \sum_{k_1=1}^{N} \sum_{i=1}^{L} \sum_{j=1}^{L} A_{k_1k_2} C_i^{\,2} \\ \cdot \Bigg\{ \left[\frac{\alpha_i + \alpha_j + 2a_{k_1}}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_1})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})}\right] \\ \cdot \frac{1}{s - (a_{k_1} + a_{k_2})} - \left[\frac{1}{(\alpha_j + a_{k_1})(\alpha_i + a_{k_2})}\right] \\ \cdot \frac{1}{s + (\alpha_j - a_{k_2})} - \left[\frac{1}{(\alpha_i + a_{k_1})(\alpha_j + a_{k_2})}\right] \\ \cdot \frac{1}{s + (\alpha_i - a_{k_2})} \\ + \left[\frac{\alpha_i + \alpha_j + 2a_{k_2}}{(\alpha_j + a_{k_2})(\alpha_i + a_{k_2})(\alpha_i + \alpha_j + a_{k_1} + a_{k_2})}\right] \\ \cdot \frac{1}{s + \alpha_i + \alpha_j} \Bigg\} \end{aligned} \tag{158}$$

If the response obtained is $y_1(t) + y_2(t)$, then $Y_1(s) + Y_2(s)$ contains poles at $s = a_{k_2}$, $k_2 = 1, \ldots, N$ whose residues are of the form

$$\left(\frac{R_j C_i}{\alpha_i + a_j}\right) + \frac{(\alpha_i + \alpha_k)\, A_{N+1,j}\, C_i^{\,2}}{(\alpha_k)\,(\alpha_i)\,(\alpha_i + \alpha_k + a_j)}$$

The pole of $Y_1(s) + Y_2(s)$ at $s = -\alpha_i$ has a residue given by

$$\sum_{j=1}^{N} \frac{R_j C_i}{\alpha_i + a_j}$$

If L = N, then there are N equations of the form

$$\begin{aligned}
\frac{R_1 C_1}{\alpha_1 + a_1} + \frac{R_2 C_1}{\alpha_1 + a_2} + \cdots + \frac{R_N C_1}{\alpha_1 + a_N} &= \beta_1 \\
\vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \qquad & \quad \vdots \\
\frac{R_1 C_N}{\alpha_N + a_1} + \frac{R_2 C_N}{\alpha_N + a_2} + \cdots + \frac{R_N C_N}{\alpha_N + a_N} &= \beta_N
\end{aligned} \tag{159}$$

or in matrix form

$$\begin{bmatrix}
\frac{C_1}{\alpha_1 + a_1} & \frac{C_1}{\alpha_1 + a_2} & \cdots & \frac{C_1}{\alpha_1 + a_N} \\
\frac{C_2}{\alpha_2 + a_1} & \frac{C_2}{\alpha_2 + a_2} & \cdots & \frac{C_2}{\alpha_2 + a_N} \\
\vdots & & & \\
\frac{C_N}{\alpha_N + a_1} & \frac{C_N}{\alpha_N + a_2} & \cdots & \frac{C_N}{\alpha_N + a_N}
\end{bmatrix}
\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}
=
\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} \tag{160}$$

or $[A]\ [R] = [\beta]$ in matrix form.

For linear independence, it is necessary that

$$\det [A] \neq 0 \tag{161}$$

This has been shown in Reference 3 provided

$$a_i \neq a_j \text{ for any } i,\ j = 1,\ldots,N;\ i \neq j$$

$$\alpha_i \neq \alpha_j \text{ for any } i,\ j = 1,\ldots,N;\ i \neq j$$

as is the case for the identification technique.

Although the procedure is more complicated, the linear and second-order impulse responses can be identified even if the second-order response cannot be isolated from the linear response. The practical application of this procedure may be complicated by the need to determine β. This is a potential numerical accuracy problem. However, in theory at least, the linear response need not be isolated from the second-order response.

Another complication in the identification procedure arises if the second-order response cannot be isolated from the third-order response. Assume that the linear impulse response of a nonlinear system has been identified. If the system cannot be excited such that the response $y_1(t) + y_2(t)$ is obtained, then the identification technique will use the response $y_1(t) + y_2(t) + y_3(t)$, where it is assumed that fourth and higher order responses are negligible compared to third order. For convenience, we define

$$y_b(t) = y_2(t) + y_3(t) \tag{162}$$

and

$$Y_b(s) = Y_2(s) + Y_3(s) \tag{163}$$

where $y_1(t)$ has been subtracted out using knowledge of $H_1(s)$ and $h_1(t)$. The poles of $Y_b(s)$ are given by

$$s = a_{k_1} + a_{k_2} + a_{k_3};\ k_1 = 1,\ldots,J;\ k_2 = 1,\ldots,M;\ k_1 = 1,\ldots,N$$

$$s = -\alpha_i - \alpha_k + a_{k_3};\ i,k = 1,\ldots,L;\ k_3 = 1,\ldots,N$$

$$s = -\alpha_i - \alpha_j - \alpha_k;\ i,j,k = 1,\ldots,L$$

$$s = a_{k_1} + a_{k_2};\ k_1 = 1,\ldots M;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i + a_{k_2};\ i = 1,\ldots,L;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i - \alpha_j;\ i,j = 1,L$$

$$s = -\alpha_i + a_{k_2} + a_{k_3};\ i,k_3 = 1,N;\ k_2 = 1,\ldots,M \qquad (164)$$

All of these quantities are known, since it was assumed that $h_1(t)$ has been identified. The problem is to identify the $A_{k_1k_2}$ of $h_2(t_1,t_2)$. The poles of $Y_b(s)$ which contain information about the $A_{k_1k_2}$ are given by

$$s = a_{k_1} + a_{k_2};\ k_1 = 1,\ldots,M;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i + a_{k_2};\ i = 1,\ldots,L;\ k_2 = 1,\ldots,N$$

$$s = -\alpha_i - \alpha_j;\ i,j = 1,\ldots,L$$

The poles of $Y_2(s)$ that correspond to poles of $Y_3(s)$ are those for which

$$a_{k_1} + a_{k_2} + a_{k_3} = a_{k_4} + a_{k_5}$$

$$k_1 = 1,\ldots,J;\ k_2 = 1,\ldots,M;\ k_1 = 1,\ldots N;\ k_4 = 1,\ldots,M$$

$$k_5 = 1,\ldots,N$$

and

$$a_{k_1} + a_{k_2} = a_{k_4};\ k_2,k_4 = 1,\ldots,N;\ k_1 = 1,\ldots,M \qquad (166)$$

These correspond to poles of the form

$$\lambda_j \qquad j = 1,\ldots,N$$

$$\lambda_\ell + \lambda_j \qquad \ell,j = 1,\ldots,N$$

$$-\alpha_i + \lambda_j \qquad i,j = 1,\ldots,N \qquad (167)$$

If the residues of these poles are known, then a set of equations exists that involves pairs of $A_{k_1k_2}$ and $C_{k_1k_2k_3}$ quantities.

The key question is whether the $A_{k_1k_2}$ and $C_{k_1k_2k_3}$ quantities can be determined.

Suppose we consider the pole at $s = \lambda_1 + \lambda_2$. The residues involve the following quantities

$$A_{12},\ C_{1,2,1},\ C_{1,2,2},\ldots,C_{1,2,N}$$

This implies that there are N + 1 unknown coefficients in the residue equation for the pole at $s = \lambda_1 + \lambda_2$. It is necessary to determine if these quantities can be solved for using the residues of $Y_2(s) + Y_3(s)$. It is shown in Reference 3 that the $C_{k_1k_2k_3}$ quantities of the form C_{iij}, C_{iji}, C_{jii}, $i,j = 1,\ldots,N$; $i \neq j$, are identified from the poles of $Y_3(s)$ given by $s = \lambda_i + \lambda_j + \lambda_k$; $i,j,k = 1,\ldots,N$; $i \neq j$, $j \neq k$. Since these poles are unique to $Y_3(s)$, the above C_{iij}, C_{iji}, and C_{jii} quantities can be determined in the usual manner. Once these are known, the A_{ij} quantities are found directly from the residue at $s = \lambda_1 + \lambda_2$. Similarly, all the $A_{k_1k_2}$, $k_1, k_2 = 1,\ldots,N$ $k_1 \neq k_2$, can be found from the poles of $Y_2(s) + Y_3(s)$ at $s = \lambda_{k_1} + \lambda_{k_2}$.

In Reference 3, it is shown that the $C_{k_1k_2k_3}$ are identified from residues of the poles at

$$\begin{aligned}
&s = 3\lambda_i && i = 1,\ldots,N \\
&s = 2\lambda_i + \lambda_j && i,j = 1,\ldots,N;\ i \neq j \\
&s = \lambda_i + \lambda_j + \lambda_k && i,j,k = 1,\ldots,N;\ i \neq j \neq k \\
&s = -\alpha_i + \lambda_j + \lambda_k && i \text{ fixed},\ j \neq k;\ j,k = 1,\ldots,N \\
&s = -\alpha_i + \lambda_j && j = 1,\ldots,N
\end{aligned} \tag{168}$$

All of these poles can be used to identify the $C_{k_1k_2k_3}$ as is normally done except for the poles at $s = -\alpha_i + \lambda_j$. We now consider the portion of the response $Y_2(s) + Y_3(s)$ due to the pole at $s = -\alpha_i + \lambda_j$. The unknown $C_{k_1k_2k_3}$ quantities are of the form C_{k_1mn} where $k_1>N$, $m>N$, $n<N$ and m and n are such that

$$a_{m_i} + a_{n_i} = \lambda_1. \tag{169}$$

The unknown $A_{k_1k_2}$ quantities are of the form $A_{k_11}, A_{k'_12},\ldots A_{k_1N_N}$ where $k_1 > N$ and $a_{k_1} + a_j = \lambda_\ell$. There are N pairs of (a_{m_i}, a_{n_i}) such that $a_{m_i} + a_{n_i} = \lambda_1$. The portion of the third order response due to the pole $s = -\alpha_1 + \lambda_1$ is given by

$$Y_{3c}(s) = \sum_{k_1=N+1}^{J} \left\{ \frac{C_{k_1 m_1 n_1}}{\alpha_1 + a_{k_1}} \sum_{i=1}^{L} \sum_{j=1}^{L} \left[-3 \left(\frac{\alpha_i + \alpha_j + 2a_{m_1}}{(\alpha_i + a_{m_1})(\alpha_j + a_{m_1})} \right) \right. \right.$$

$$\left. \cdot \left(\frac{1}{\alpha_i + \alpha_j + a_{m_1} + a_{n_1}} \right) \right] + \frac{C_{k_1 m_2 n_2}}{\alpha_1 + a_{k_1}} \sum_{i=1}^{L} \sum_{j=1}^{L} \left[-3 \left(\frac{\alpha_i + \alpha_j + 2a_{m_2}}{(\alpha_i + a_{m_2})(\alpha_j + a_{m_2})} \right) \right.$$

$$\left. \cdot \left(\frac{1}{\alpha_i + \alpha_j + a_{m_2} + a_{n_2}} \right) \right] + \ldots + \frac{C_{k_1 m_N n_N}}{\alpha_1 + a_{k_1}} \sum_{i=1}^{L} \sum_{j=1}^{L}$$

$$\cdot \left[-3 \left(\frac{\alpha_i + \alpha_j + 2a_{m_N}}{(\alpha_i + a_{m_N})(\alpha_j + a_{m_N})} \right) \left(\frac{1}{\alpha_i + \alpha_j + a_{m_N} + a_{n_N}} \right) \left(\frac{1}{s + \alpha_1 - \lambda_1} \right) \right] \tag{170}$$

The portion of the second-order response at $s = -\alpha_1 + \lambda_\ell$ is given by

$$Y_{2c}(s) = \sum_{k_1=1}^{M} \left[\left(\frac{A_{k_1 1}}{(\alpha_1 + a_{k_1})} \right) \sum_{i=1}^{L} \frac{1}{(\alpha_i + a_1)} + \left(\frac{A_{k_1 2}}{(\alpha_2 + a_{k_1})} \right) \right.$$

$$\left. \cdot \sum_{i=1}^{L} \frac{1}{(\alpha_i + a_2)} + \ldots + \left(\frac{A_{k_1 N}}{(\alpha_N + a_{k_1})} \right) \sum_{i=1}^{L} \frac{1}{(\alpha_i + a_N)} \right] \tag{171}$$

There are $2N^3$ unknown $C_{k_1k_2k_3}$ quantities and N^2 unknown $A_{k_1k_2}$ quantities in the $Y_{2c}(s) + Y_{3c}(s)$ portion of the response.

The residue at pole $s = -\alpha_1 + \lambda_1$ is defined to be θ_1. The functional form of the residue can be written as

$$3\sum_{k_1=N+1}^{J}\left[\frac{C_{k_1m_1n_1}}{\alpha_1+a_{k_1}}\beta_1+\frac{C_{k_2m_2n_2}}{\alpha_2+a_{k_1}}\beta_2+\ldots+\frac{C_{k_1m_Nn_N}}{\alpha_N+a_{k_1}}\beta_N\right]$$

$$\sum_{k_1=N+1}^{M}\left[\frac{A_{k_11}}{\alpha_1+a_{k_1}}\gamma_1+\frac{A_{k_12}}{\alpha_2+a_{k_1}}\gamma_2+\ldots+\frac{A_{k_1N}}{\alpha_N+a_{k_1}}\gamma_N\right]=\theta_1 \tag{172}$$

This can be reduced to

$$\sum_{k_1=N+1}^{M}\left[\left(\frac{3C_{k_1m_1n_1\beta_1}+A_{k_11}\gamma_1}{\alpha_1+a_{k_1}}\right)+\left(\frac{3C_{k_2m_2n_2}\beta_2+A_{k_12}\gamma_2}{\alpha_2+a_{k_1}}\right)+\ldots\right.$$

$$\left.+\left(\frac{3C_{k_1m_Nn_N}\beta_N+A_{k_1N}\gamma_N}{\alpha_N+a_{k_1}}\right)\right]+\sum_{k_1=M+1}^{J}\left[\frac{3C_{k_1m_1n_1}\beta_1}{\alpha_1+a_{k_1}}+\frac{3C_{k_1m_2n_2}\beta_2}{\alpha_2+a_{k_1}}\right.$$

$$\left.+\ldots+\frac{3C_{k_1m_Nn_N}\beta_N}{\alpha_3+a_{k_N}}\right]=\theta_1 \tag{173}$$

Let $\varepsilon_1=\beta_1/\gamma_1$, then rewrite equation (173) as

$$\sum_{k_1=M+1}^{N}\left[\left(\frac{3C_{k_1m_1n_1}+A_{k_11}\varepsilon_1}{\alpha_1+a_{k_1}}\right)\beta_1+\ldots+\left(\frac{3C_{k_1m_1n_1}\beta_1+A_{k_1N}\varepsilon_N}{\alpha_N+a_{k_1}}\right)\beta_N\right]$$

$$\sum_{k_1=M+1}^{J}\frac{3C_{k_1m_1n_1}}{\alpha_1+a_{k_1}}\beta_1+\ldots+\frac{3C_{k_1m_1n_1}\beta_1}{\alpha_N+a_{k_1}}=\theta_1 \tag{174}$$

Furthermore, equation (173) can be rewritten as

$$\sum_{i=1}^{2N^2}\frac{F_j}{\alpha_1+f_j{}^{\alpha}}=\theta_1 \tag{175}$$

where $F_j = C_{k_1 m_i n_i}\beta_i + A_{k_i i}\varepsilon_i$ | for k_1 which have nonzero $C_{k_1 m_i n_i}$ and $\varepsilon_i = 0$ for $i > N$

A similar equation to (175) is obtained for each value of α_i, $i=1, \ldots, L$. If $L = 2N^2$, then the identification technique generates the equations

$$\sum_{j=1}^{2N^2} \frac{F_j}{\alpha_1 + f_j^{\alpha}} = \theta_1$$

$$\sum_{j+1}^{2N^2} \frac{F_j}{\alpha_2 + f_j^{\alpha}} = \theta_2 \qquad (176)$$

$$\vdots$$

$$\sum_{j=1}^{2N^2} \frac{F_j}{\alpha_{2N^2} + f_j^{\alpha}} = \theta_{2N^2}$$

The set of equations in (176) was previously shown to be linearly independent. Therefore, the set of equations provides a unique solution for the F_j quantities.

There still remain N pairs of $C_{k_1 k_2 k_3}$ and $A_{k_1 k_2}$ terms which have not been identified. These are of the form

$$3C_{k_1 m_1 n_1} + A_{k_1 1}\varepsilon_1$$

These quantities must be separated to completely identify $h_2(t_1, t_2)$ and $h_3(t_1\ t_2, t_3)$. There is a need to find a way to separately identify these $A_{k_1 k_2}$ and $C_{k_1 k_2 k_3}$ quantities.

The only source for unique identification of the $A_{k_1 k_2}$ quantities is the residues of the poles at $s = -\alpha_i - \alpha_j$. These poles are unique to $Y_2(s)$ and the residues involve only the unknown $A_{k_1 k_2}$ quantities. The problem is to demonstrate whether or not these residues can be used to generate a set of linearly independent equations to permit solution for the $A_{k_1 k_2}$. Consider the pole at $s = -2\alpha_1$. The residue is of the form

$$\left(\frac{1}{2\alpha_1 + \lambda_1}\right) \sum_{k'_1=1}^{N} \sum_{k_2=1}^{N} A_{k'_1k_2} \left[\frac{1}{(\alpha_1 + a_{k_2})}\right]$$

$$+ \left(\frac{1}{2\alpha_1 + \lambda_2}\right) \sum_{k''_1=1}^{N} \sum_{k_2=1}^{N} A_{k''_1k_2} \left[\frac{1}{(\alpha_1 + a_{k_2})}\right]$$

$$+ \dots$$

$$+ \left(\frac{1}{2\alpha_1 + \lambda_2}\right) \sum_{k_1^{N'}=1}^{N} \sum_{k_2=1}^{N} A_{k_1^{N'}k_2} \left(\frac{1}{\alpha_1 + a_{k_2}}\right)$$

where k_1', k_1'', ... $k_1^{N'}$ correspond to those values of k_1 for which $A_{k_1k_2}$ are unknown. This equation involves N^2 unknown $A_{k_1k_2}$ quantities. Analysis has not been able to show that this set of equations is linearly independent or dependent. The equations of interest for linear independence become

$$\frac{1}{2\alpha_1 + a_1} \sum_{i=1}^{N} \frac{C_i}{\alpha_1 + a_i} + \frac{1}{2\alpha_1 + a_2} \sum_{i=N+1}^{2N} \frac{C_i}{\alpha_1 + a_i}$$

$$+ \dots + \frac{1}{2\alpha_1 + a_N} \sum_{i=N^2-N+1}^{N^2} \frac{C_i}{\alpha_1 + a_i} = 0$$

$$\frac{1}{2\alpha_2 + a_1} \sum_{i=1}^{N} \frac{C_i}{\alpha_2 + a_i} + \frac{1}{2\alpha_2 + a_2} \sum_{i=N}^{2N} \frac{C_i}{\alpha_2 + a_i}$$

$$+ \dots + \frac{1}{2\alpha_2 + a_N} \sum_{i=N^2-N+1}^{N^2} \frac{C_i}{\alpha_1 + a_i} = 0$$

$$\vdots$$

$$\frac{1}{2\alpha_N + a_1} \sum_{i=1}^{N} \frac{C_i}{\alpha_N + a_N} + \frac{1}{2\alpha_2 + a_2} \sum_{i=N+1}^{2N} \frac{C_i}{\alpha_N + a_i}$$

$$+ \dots + \frac{1}{2\alpha_N + a_N} \sum_{i=N^2-N+1}^{N^2} \frac{C_i}{\alpha_N + a_i} = 0 \qquad (177)$$

If this set of equations implies $C_i=0$, $i=1, \ldots, N^2$, then the set of equations involving the $A_{k_1k_2}$ is linearly independent. Successive solution of these simultaneous equations does not produce a factorable polynomial to demonstrate independence. The resultant equation cannot be solved and linear dependence or independence cannot be shown. A similar situation exists for the unknown $C_{k_1k_2k_3}$ quantities and the poles at $s = -\alpha_i - \alpha_j - \alpha_k$ that are unique to $Y_3(s)$.

This analysis has failed to demonstrate that there is no need to isolate the second-order system response from the third-order response. Therefore, it is concluded that, for a practical implementation of the identification technique, it is necessary to excite the nonlinear system so that the third order and higher order responses are negligible compared to the linear and second order response.

G. ALTERNATIVE IDENTIFICATION PROCESSING ALGORITHMS

The identification technique described in this report is based on the pencil of functions approach to linear system identification. The first step in the nonlinear identification process is the identification of the poles and residues of the linear system transfer function. The pencil-of-functions approach is well-suited to accomplishing this identification. The second step in the nonlinear system identification process is the identification of the residues of the poles of $Y_2(s)$. These poles of $Y_2(s)$ are known once the linear transfer function is identified. The pencil-cf-functions approach is still used, but it must be noted that other potential methods of identifying the residues of the poles of $Y_2(s)$ exist that might alleviate some of the computational complexity associated with the pencil-of-functions approach.

The basic problem at this point is to identify the R_k quantities in the equation

$$y_2(t) = \sum_{k=1}^{\beta} R_k \, e^{-\varepsilon_k t} \tag{178}$$

where $y_2(t)$ is the second order response of a nonlinear system. In this equation, the ε_k are known, since they are related to the poles of the linear transfer function of the system. A sampled time history of $y_2(t)$ is obtained via measurement of the output of the nonlinear system. The objective then is to use this information to evaluate the R_k.

A detailed review of several candidate algorithms for solving this problem has been addressed in Reference 6. These identification approaches include the least-squares method, orthonormal least squares method, equality of derivatives method, equality of integrals method and the generalized integrated squared error. Details of these approaches are given in Reference 6 and are not repeated here.

The basic approach used in many of these methods is to sample the time function (in this case, $y_2(t)$) M times where $M > \beta$ [β is the number of poles in $Y_2(s)$ or natural frequencies in $y_2(t)$]. Then, each technique attempts to minimize an error function to determine an "optimum" set of R_k coefficients. In general these techniques require inversion of a $\beta \times \beta$ matrix to determine the R_k coefficients. The matrix entries involve functions of the $e^{-\epsilon_k t}$ and in this sense are very similar to the technique used in the pencil-of-functions approach. The advantage of the pencil-of-functions method is that no approximations are used as is the case with these overdetermined system approaches ($M > \beta$). Because of this advantage and the requirement that a $\beta \times \beta$ matrix be inverted by these other identification techniques, the pencil-of-functions approach appears to be as good a candidate for this nonlinear system identification technique as the others described in Reference 6.

SECTION IV

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